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On the average character degree and the average class size in finite groups ☆

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ABSTRACT

Let $\mathrm{acd}(G)$ and $\mathrm{acs}(G)$ denote the average irreducible character degree and the average conjugacy class size, respectively, of a finite group G. The main objective of this paper is to find the lower bounds of $\mathrm{acd}(G)$ and $\mathrm{acs}(G)$ for a non-r-solvable group G where r is a prime. We show that $PSL(2, r)$, where $r \geq 5$, is a non-r-solvable group with the smallest average character degree and the smallest average class size.

1. Introduction

In this paper, G always denotes a finite group, all characters are complex characters, and we use Isaacs [13] as a source for standard notation and results from character theory. We write

$$\mathrm{acd}(G) = \frac{\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)}{|\mathrm{Irr}(G)|},$$

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so that $\mathrm{acd}(G)$ is the average irreducible character degree of G. K. Magaard and H.P. Tong-Viet proved in [16, Theorem 1.4] that if $\mathrm{acd}(G) \leq 2$, then G is solvable. Recently, I.M. Isaacs, M. Loukaki and A. Moreto [14] improved the result and showed that if $\mathrm{acd}(G) \leq 3$ then G is solvable. They also conjectured that G is solvable whenever $\mathrm{acd}(G) < 16/5 = \mathrm{acd}(PSL(2,5))$.

Recall that there is often an analogy between results about character degrees and those about conjugacy class sizes. Denote by $k(G)$ the number of conjugacy classes of G. We have $k(G) = |\mathrm{Irr}(G)|$. In particular, writing $\mathrm{acs}(G)$ to denote the average class size of a finite group G, we have

$$\mathrm{acs}(G) = \frac{|G|}{k(G)} = \frac{\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2}{|\mathrm{Irr}(G)|} \geq \frac{(1/|\mathrm{Irr}(G)|)(\sum_{\chi\in\mathrm{Irr}(G)}\chi(1))^2}{|\mathrm{Irr}(G)|} = \big(\mathrm{acd}(G)\big)^2,$$

where the inequality holds by Cauchy–Schwarz. As shown in [11] and [9], $(\mathrm{acs}(G))^{-1}$ is just the probability that a randomly chosen pair of elements of G commute. There are many results about $\mathrm{acs}(G)$ or $(\mathrm{acs}(G))^{-1}$ (see [11,9,10]). For example, it is known that G is solvable whenever $\mathrm{acs}(G) < 12 = \mathrm{acs}(PSL(2,5))$ (see [9,10]).

Let r be a prime divisor of $|G|$. The main objective of this paper is to find the lower bounds of $\mathrm{acd}(G)$ and $\mathrm{acs}(G)$ for a non-r-solvable group G. We will show that $PSL(2,r)$, where $r \geq 5$, is a non-r-solvable group with the smallest average character degree and the smallest average class size.

Let r be a prime. We define

$$f(r) = \begin{cases} (r^2+r+2)/(r+5), & \text{if } r \equiv 1 \pmod 4, \\ (r^2+r)/(r+5), & \text{if } r \geq 7 \text{ and } r \equiv 3 \pmod 4, \\ 16/5 = f(5), & \text{if } r = 2,3, \end{cases}$$

and

$$g_0(r) = \begin{cases} r(r^2-1)/(r+5), & \text{if } r \geq 5, \\ 12 = g_0(5), & \text{if } r = 2,3, \end{cases} \qquad g(r) = \begin{cases} r(r^2-1)/(r+4), & \text{if } r \geq 5, \\ 40/3 = g(5), & \text{if } r = 2,3. \end{cases}$$

Note that for $r \geq 5$, we have (see Lemma 2.4)

$$f(r) = \mathrm{acd}\big(PSL(2,r)\big), \qquad g_0(r) = \mathrm{acs}\big(PSL(2,r)\big), \qquad g(r) = \mathrm{acs}\big(SL(2,r)\big).$$

The following facts about $f(r)$, $g_0(r)$ and $g(r)$ are elementary, and will be used freely in the proofs.

(a) If $r \geq 5$, then $r-1 > \sqrt{g(r)} > \sqrt{g_0(r)} > f(r)$; and if $r \geq 7$, then $r-2 > f(r)$.
(b) For primes r_1, r_2 with $r_1 > r_2 \geq 5$, $f(r_1) > f(r_2)$, $g(r_1) > g(r_2)$, $f(r_1) > \sqrt{g(r_2)}$.
(c) $2f(r) > \sqrt{g(r)}$.