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Computations for Coxeter arrangements and Solomon's descent algebra III: Groups of rank seven and eight

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A B S T R A C T

In this paper we extend the computations in parts I and II of this series of papers and complete the proof of a conjecture of Lehrer and Solomon expressing the character of a finite Coxeter group W acting on the p-th graded component of its Orlik–Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of W for groups of rank seven and eight. For classical Coxeter groups, these characters are given using a formula that is expected to hold in all ranks.

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1. Introduction

Suppose that W is a finite Coxeter group and that V is the complexified reflection representation of W. Let $\mathcal{A}$ be the set of reflecting hyperplanes of W in V and let

$$M = V \setminus \bigcup_{H \in \mathcal{A}} H$$

denote the complement of these hyperplanes in V. The reflection length of an element $w \in W$ is the least integer p such that w may be written as a product of p reflections. Clearly, conjugate elements have the same reflection length. Lehrer and Solomon [16, 1.6] conjectured that there is a $\mathbb{C}W$-module isomorphism

$$H^p(M, \mathbb{C}) \cong \bigoplus_c \operatorname{Ind}_{C_W(c)}^W \chi_c, \quad p = 0, \ldots, \operatorname{rank}(W) \tag{1.1}$$

where c runs over a set of representatives of the conjugacy classes of W with reflection length equal to p and χ_c is a suitable linear character of the centralizer $C_W(c)$ of c in W. Lehrer and Solomon proved (1.1) for symmetric groups. In [8] we proposed an inductive approach to the Lehrer–Solomon conjecture that establishes a direct connection between the character of the Orlik–Solomon algebra of W and the regular character of W. This inductive approach has been used to prove (1.1) for dihedral groups [8], symmetric groups [7], and Coxeter groups with rank at most six [2,3].

In this paper we extend the computations in [2,3] to finite Coxeter groups of rank seven and eight and complete the proof of the conjectures in [8] that relate the Orlik–Solomon and regular characters of these groups. As a consequence, the conjectures in [8], as well as the Lehrer–Solomon conjecture, are shown to hold for all finite Coxeter groups of rank at most eight. In particular, these conjectures hold for all exceptional finite Coxeter groups.

As described in more detail below, the proof for the Coxeter groups of type E_7 and E_8 uses the techniques developed in [2,3], except that we use Fleischmann and Janiszczak's computation [10] of the Möbius functions of fixed point sets in the intersection lattice of $\mathcal{A}$ to compute the character of the top component of the Orlik–Solomon algebra. We take this opportunity to correct several minor misprints in the table for E_8 in [10]. For Coxeter groups of classical types in this paper we give explicit formulas for the characters χ_c in all ranks and then use the methods developed in [2,3] to verify that (1.1) holds for rank less than or equal to eight. The formulas for the characters χ_c given below are similar to the formulas in [16].

The rest of this paper is organized as follows. In Section 2 we establish notation, give a concise review of the constructions and conjectures in [2,3,8], and state the main theorem to be proved in this paper (Theorem 1); in Section 3 we give explicit constructions of the linear characters that we expect to satisfy the conclusion of Theorem 1 for classical groups, and we verify that these characters do indeed satisfy the conclusion of Theorem 1