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Superpotentials, Calabi–Yau algebras, and PBW deformations

J. Karmazyn

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ABSTRACT

The paper [9] by Bocklandt, Schedler and Wemyss considers path algebras with relations given by the higher derivations of a superpotential, giving a condition for such an algebra to be Calabi–Yau. In particular they show that the algebra $\mathbb{C}[V]\rtimes G$, for V a finite dimensional $\mathbb{C}$ vector space and G a finite subgroup of $\mathrm{GL}(V)$, is Morita equivalent to a path algebra with relations given by a superpotential, and is Calabi–Yau for $G < \mathrm{SL}(V)$. In this paper we extend these results, giving a condition for a PBW deformation of a Calabi–Yau, Koszul path algebra with relations given by a superpotential to have relations given by a superpotential, and proving these are Calabi–Yau in certain cases.

We apply our methods to symplectic reflection algebras, where we show that every symplectic reflection algebra is Morita equivalent to a path algebra whose relations are given by the higher derivations of an inhomogeneous superpotential. In particular we show these are Calabi–Yau regardless of the deformation parameter.

Also, for G a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ not contained in $\mathrm{SL}_2(\mathbb{C})$, we consider PBW deformations of a path algebra with relations which is Morita equivalent to $\mathbb{C}[x,y]\rtimes G$. We show there are no non-trivial PBW deformations when G is a small subgroup.

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1. Introduction

1.1. Introduction

In this paper we consider path algebras of quivers with certain relations, in particular studying relations produced from a superpotential. Given a quiver Q, a homogeneous superpotential of degree n is an element, $\Phi_n = \sum c_{a_1 \dots a_n} a_1 \dots a_n$, in the path algebra of Q satisfying the n *superpotential condition*: $c_{aq} = (-1)^{n-1} c_{qa}$ for all arrows a and paths q. From such a superpotential Φ_n and a non-negative integer k we construct an algebra $\mathcal{D}(\Phi_n, k) := \frac{\mathbb{C}Q}{R}$ as a path algebra with relations R. These relations are constructed by the process of *differentiation*, where we define the left derivative of a path p by a path q, denoted $\delta_q p$, to be t if $p = qt$ and 0 otherwise, and the relations are given by $R = \langle \{\delta_p \Phi_n : |p| = k\} \rangle$.

Algebras of this form are considered in [9], where they are related to Calabi–Yau (CY), N-Koszul algebras. In [9] a complex, $\mathcal{W}^\bullet$, is defined which depends only on the superpotential, and a path algebra with relations is N-Koszul and CY if and only if it is of the form $\mathcal{D}(\Phi_n, k)$ for a superpotential Φ_n and $\mathcal{W}^\bullet$ is a resolution.

Skew group algebras, $\mathbb{C}[V] \rtimes G$, for G a finite subgroup of $\mathrm{GL}(V)$, are Morita equivalent to path algebras of this form. These are 2-Koszul, and CY when $G < \mathrm{SL}(V)$, and hence their relations can be given by a superpotential. An explicit way to calculate this superpotential is given in [9, Theorem 3.2].

We prove two results concerning the PBW deformations of $(n-k)$-Koszul, $(k+2)$-CY algebras of the form $\mathcal{D}(\Phi_n, k)$. We define an inhomogeneous superpotential of degree n to be an element of the path algebra $\Phi' := \Phi_n + \phi_{n-1} + \ldots + \phi_k$, such that each $\phi_j := \sum c_p p$ is a sum of elements of the path algebra of length j, and each ϕ_j satisfies the n superpotential condition. Such a superpotential defines relations $P = \langle \{\delta_p \Phi' : |p| = k\} \rangle$,