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# Superpotentials, Calabi–Yau algebras, and PBW deformations



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## ABSTRACT

The paper [9] by Bocklandt, Schedler and Wemyss considers path algebras with relations given by the higher derivations of a superpotential, giving a condition for such an algebra to be Calabi–Yau. In particular they show that the algebra  $\mathbb{C}[V] \rtimes G$ , for  $V$  a finite dimensional  $\mathbb{C}$  vector space and  $G$  a finite subgroup of  $\mathrm{GL}(V)$ , is Morita equivalent to a path algebra with relations given by a superpotential, and is Calabi–Yau for  $G < \mathrm{SL}(V)$ . In this paper we extend these results, giving a condition for a PBW deformation of a Calabi–Yau, Koszul path algebra with relations given by a superpotential to have relations given by a superpotential, and proving these are Calabi–Yau in certain cases.

We apply our methods to symplectic reflection algebras, where we show that every symplectic reflection algebra is Morita equivalent to a path algebra whose relations are given by the higher derivations of an inhomogeneous superpotential. In particular we show these are Calabi–Yau regardless of the deformation parameter.

Also, for  $G$  a finite subgroup of  $\mathrm{GL}_2(\mathbb{C})$  not contained in  $\mathrm{SL}_2(\mathbb{C})$ , we consider PBW deformations of a path algebra with relations which is Morita equivalent to  $\mathbb{C}[x, y] \rtimes G$ . We show there are no non-trivial PBW deformations when  $G$  is a small subgroup.

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## Contents

1. Introduction .....	101
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1.1.	Introduction . . . . .	101
1.2.	Main results . . . . .	102
1.3.	Contents . . . . .	103
2.	Preliminaries . . . . .	103
2.1.	Quivers and superpotentials . . . . .	103
2.2.	Calabi–Yau algebras . . . . .	107
2.3.	Koszul algebras . . . . .	108
2.4.	Superpotentials and higher order derivations . . . . .	109
2.5.	PBW deformations . . . . .	111
3.	Main results . . . . .	113
3.1.	Deformations of superpotential algebras . . . . .	114
3.2.	CY property of deformations . . . . .	117
4.	Application: symplectic reflection algebras . . . . .	122
4.1.	Symplectic reflection algebras as superpotential algebras . . . . .	123
4.1.1.	Examples . . . . .	124
5.	Application: PBW deformations of skew group rings for $GL_2$ . . . . .	127
	Acknowledgments . . . . .	130
	Appendix A. McKay quivers for finite small subgroups of $GL_2(\mathbb{C})$ . . . . .	130
	References . . . . .	134

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## 1. Introduction

### 1.1. Introduction

In this paper we consider path algebras of quivers with certain relations, in particular studying relations produced from a superpotential. Given a quiver  $Q$ , a homogeneous superpotential of degree  $n$  is an element,  $\Phi_n = \sum c_{a_1 \dots a_n} a_1 \dots a_n$ , in the path algebra of  $Q$  satisfying the  $n$  *superpotential condition*:  $c_{aq} = (-1)^{n-1} c_{qa}$  for all arrows  $a$  and paths  $q$ . From such a superpotential  $\Phi_n$  and a non-negative integer  $k$  we construct an algebra  $\mathcal{D}(\Phi_n, k) := \frac{\mathbb{C}Q}{R}$  as a path algebra with relations  $R$ . These relations are constructed by the process of *differentiation*, where we define the left derivative of a path  $p$  by a path  $q$ , denoted  $\delta_q p$ , to be  $t$  if  $p = qt$  and 0 otherwise, and the relations are given by  $R = \langle \{\delta_p \Phi_n : |p| = k\} \rangle$ .

Algebras of this form are considered in [9], where they are related to Calabi–Yau (CY),  $N$ -Koszul algebras. In [9] a complex,  $\mathcal{W}^\bullet$ , is defined which depends only on the superpotential, and a path algebra with relations is  $N$ -Koszul and CY if and only if it is of the form  $\mathcal{D}(\Phi_n, k)$  for a superpotential  $\Phi_n$  and  $\mathcal{W}^\bullet$  is a resolution.

Skew group algebras,  $\mathbb{C}[V] \rtimes G$ , for  $G$  a finite subgroup of  $GL(V)$ , are Morita equivalent to path algebras of this form. These are 2-Koszul, and CY when  $G < SL(V)$ , and hence their relations can be given by a superpotential. An explicit way to calculate this superpotential is given in [9, Theorem 3.2].

We prove two results concerning the PBW deformations of  $(n-k)$ -Koszul,  $(k+2)$ -CY algebras of the form  $\mathcal{D}(\Phi_n, k)$ . We define an inhomogeneous superpotential of degree  $n$  to be an element of the path algebra  $\Phi' := \Phi_n + \phi_{n-1} + \dots + \phi_k$ , such that each  $\phi_j := \sum c_p p$  is a sum of elements of the path algebra of length  $j$ , and each  $\phi_j$  satisfies the  $n$  superpotential condition. Such a superpotential defines relations  $P = \langle \{\delta_p \Phi' : |p| = k\} \rangle$ ,

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