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Symbolic powers of perfect ideals of codimension 2 and birational maps

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A R T I C L E I N F O

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To Wolmer Vasconcelos on his 75th birthday, for his seminal mathematical ideas

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A B S T R A C T

This work is about symbolic powers of codimension two perfect ideals in a standard polynomial ring over a field, assuming that the entries of the corresponding presentation matrix are general linear forms. The main contribution of the present approach is the use of the birational theory underlying the nature of the ideal and the details of a deep interlacing between generators of its symbolic powers and the inversion factors stemming from the inverse map to the birational map defined by the linear system spanned by the generators of this ideal. A full description of the corresponding symbolic Rees algebra is given in some cases. One application is an affirmative solution of a conjecture of Eisenbud–Mazur in [11, Section 2].

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Birational map
Inversion factor

Introduction

Let $I \subset R$ denote an ideal in a Noetherian ring and let $r \geq 0$ be an integer. The rth symbolic power $I^{(r)}$ of I can be defined as the inverse image of $U^{-1}I^r$ under the natural homomorphism $R \to U^{-1}R$ of fractions, where U is the complementary set of the union of the associated primes of R/I. There is a known hesitation as to whether one should take the whole set of associated primes of R/I or just its minimal primes or even those of minimal codimension or maximal dimension. In this work we need not worry about this dilemma because the notion will only be employed in the case of a codimension 2 perfect ideal in a Cohen–Macaulay ring — actually, a polynomial ring over a field. In this setup there is no ambiguity and $I^{(r)}$ is precisely the intersection of the primary components of the ordinary power I^r relative to the associated primes of R/I, i.e., the unmixed part of I^r.

A more serious problem is the characteristic of the base field. In characteristic zero, if I is a radical ideal, one has the celebrated Zariski–Nagata differential characterization of $I^{(r)}$ (see [9, 3.9] and the references there). The subject in positive characteristic or mixed characteristic gives a quite different panorama, often much harder but with different methods anyway. Essential parts of this work assume characteristic zero. This is not due to a need of using the Zariski–Nagata criterion upfront, but rather to an urge of dealing with Jacobian matrices and using Bertini's theorem. Many technical results will be valid just over an infinite field, hence there has been an effort to convey when the characteristic is an issue at specific places. On the other hand, since we will draw quite substantially on aspects of birational maps, it may be a good idea in those instances to think about k as being algebraically closed.

The main object of concern is an $m \times (m-1)$ matrix whose entries are general 1-forms in a polynomial ring $R = k[X_1, \ldots, X_n]$ over an infinite field k — called herein *general linear matrices*. We will focus on the ideal $I \subset R$ generated by the $(m-1)$-minors of the matrix. Knowingly, the group $\mathrm{Gl}(m,k) \times \mathrm{Gl}(n,k) \times \mathrm{Gl}(m-1,k)$ acts on the set of all linear $m \times (m-1)$ matrices over k. An important notion regarding these matrices is that of being 1-generic in the sense of [8, Proposition–Definition 1.1]. Unfortunately, an $m \times (m-1)$ linear matrix is 1-generic only if $n \geq 2(m-1)$ [8, Proposition 1.3]. Thus, 1-genericity covers so to say only "half" the cases. In particular, for $n < 2(m-1)$, the above triple action does not preserve the property of being general linear, as is clear that one may introduce a certain number of zero entries in the matrix, up to elementary k-linear row-column operations. The property of being a general linear matrix is however preserved if the action is restricted to a suitable open set of $\mathrm{Gl}(m,k) \times \mathrm{Gl}(n,k) \times \mathrm{Gl}(m-1,k)$. As a simple example, take $m = 2$, $n = 1$. Then the 2×1 matrix $(\alpha x, \beta x)^t (\alpha \neq 0)$ can be converted to $(\alpha x, 0)^t$ by the left action of the element