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# Simple, locally finite dimensional Lie algebras in positive characteristic

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We prove two structure theorems for simple, locally finite dimensional Lie algebras over an algebraically closed field of characteristic $p$ which give sufficient conditions for the algebras to be of the form $[R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])$ or $[K(R,*), K(R,*)]$ for a simple, locally finite dimensional associative algebra $R$ with involution $*$. The first proves that a condition we introduce, known as locally nondegenerate, along with the existence of an ad-nilpotent element suffice. The second proves that an ad-integrable Lie algebra is of this type if the characteristic of the ground field is sufficiently large. Lastly we construct a simple, locally finite dimensional associative algebra $R$ with involution $*$ such that $K(R,*) \neq [K(R,*), K(R,*)]$ to demonstrate the necessity of considering the commutator in the first two theorems.



## 1. Introduction

An important result which characterizes the simple, finite dimensional Lie algebras over a field of positive characteristic is the Kostrikin–Strade–Benkart Theorem [20,25,7]: Suppose $L$ is a simple, finite dimensional Lie algebra over an algebraically closed field of

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characteristic $p > 5$. If $L$ is nondegenerate and there exists a nonzero element $x \in L$ such that $ad(x)^{p-1} = 0$, then $L$ is a Lie algebra of classical type. This theorem was improved by Premet in [23]: Every finite dimensional, nondegenerate simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is classical.

Suppose $L$ is a simple, infinite dimensional Lie algebra over an algebraically closed field of characteristic zero. In [1], Bahturin, Baranov, and Zalesskii prove that $L$ embeds into a locally finite associative algebra if and only if $L$ is isomorphic to $[K(R, *), K(R, *)]$ where $*$ is an involution and $R$ is an involution simple locally finite associative algebra. This utilizes and extends earlier work of Baranov in [5]. In [26], Zalesskii asks for a characterization of such Lie algebras when the ground field is algebraically closed of positive characteristic. The following result extends the Kostrikin–Strade–Benkart Theorem to the infinite dimensional case and also addresses Zalesskii's question in the spirit of this theorem.

**Theorem 1.** *Let $L$ be a simple, infinite dimensional, locally finite Lie algebra over an algebraically closed field $F$ of characteristic $p > 7$ or characteristic zero. Then the following conditions are equivalent:*

1. *$L$ is locally nondegenerate and there is some nonzero element $x \in L$ such that $ad(x)^{p-1} = 0$.*
2. *$L \cong [R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$.*

For the definition of locally nondegenerate, see Section 2.4. We note that over a field of characteristic zero, the condition that $L$ be locally nondegenerate is trivial. Also, when the characteristic of the ground field is zero, the condition that $ad(x)^{p-1} = 0$ for some $x$ simply means that there exists some ad-nilpotent element (the index of nilpotence is inconsequential).

Another result of this kind does not assume $L$ is locally nondegenerate, but imposes a stronger assumption on the characteristic of the ground field. We say $L$ is *ad-integrable* (or has an *algebraic adjoint representation*, see [28]) if for each element $a \in L$, $ad(a)$ is a root of some nonzero polynomial $f_a(t) \in F[t]$. Note that a simple, locally finite Lie algebra embeds into a locally finite associative algebra if and only if it is ad-integrable (see for example Theorem 9.1 of [5]). For $a \in L$, let $\mu_a(t)$ be the minimal polynomial of $ad(a)$ and define $d(L) = min\{\deg \mu_a(t) \mid a \text{ is not ad-nilpotent}\}$.

**Theorem 2.** *Let $L$ be a simple, locally finite Lie algebra which is ad-integrable over an algebraically closed field $F$ of characteristic $p > 3$. If $p > 2d(L) - 2$, then $L \cong [R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$.*