Journal of Algebra 413 (2014) 320–344

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

How fast do polynomials grow on semialgebraic sets?

Pinaki Mondal[a], Tim Netzer[b,*]

[a] Weizmann Institute of Science, Israel
[b] Universität Leipzig, Germany

ARTICLE INFO

Article history:
Received 3 June 2013
Available online 5 June 2014
Communicated by Steven Dale Cutkosky

MSC:
primary 14P10, 16W50, 44A60
secondary 14M25, 14M27

Keywords:
Semialgebraic sets
Compactifications
Graded algebra
Moment problem

ABSTRACT

We study the growth of polynomials on semialgebraic sets. For this purpose we associate a graded algebra with the set, and address all kinds of questions about finite generation. We show that for a certain class of sets, the algebra is finitely generated. This implies that the total degree of a polynomial determines its growth on the set, at least modulo bounded polynomials. We, however, also provide several counterexamples, where there is no connection between total degree and growth. In the plane, we give a complete answer to our questions for certain simple sets, and we provide a systematic construction for examples and counterexamples. Some of our counterexamples are of particular interest for the study of moment problems, since none of the existing methods seems to be able to decide the problem there. We finally also provide new three-dimensional sets, for which the algebra of bounded polynomials is not finitely generated.

Contents

* Corresponding author.
E-mail addresses: pinaki@math.toronto.edu (P. Mondal), netzer@math.uni-leipzig.de (T. Netzer).

http://dx.doi.org/10.1016/j.jalgebra.2014.05.006
0021-8693/

1. Introduction

Let $\mathbb{R}[x]$ be the polynomial algebra in n variables $x = (x_1, \ldots, x_n)$. For $d \in \mathbb{N}$ we denote by $\mathbb{R}[x]_d = \{p \in \mathbb{R}[x] \mid \deg(p) \leq d\}$ the finite dimensional subspace of polynomials of total degree at most d. For a set $S \subseteq \mathbb{R}^n$ let

$$\mathcal{B}_d(S) := \{p \in \mathbb{R}[x] \mid p^2 \leq q \text{ on } S, \text{ for some } q \in \mathbb{R}[x]_{2d}\}$$

denote the set of those polynomials, that grow on S as if they were of degree at most d. Clearly $\mathbb{R}[x]_d \subseteq \mathcal{B}_d(S)$ for all d, and $\mathcal{B}_d(S)$ is closed under addition, which follows for example from the inequality

$$(p + p')^2 \leq (p + p')^2 + (p - p')^2 = 2p^2 + 2p'^2.$$

$\mathcal{B}_0(S)$ is the algebra of *bounded polynomials* on S, and each $\mathcal{B}_d(S)$ carries the structure of a $\mathcal{B}_0(S)$-module. More generally, $\mathcal{B}_d(S) \cdot \mathcal{B}_{d'}(S) \subseteq \mathcal{B}_{d+d'}(S)$, so we have a graded algebra

$$\mathcal{B}(S) := \bigoplus_{d \geq 0} \mathcal{B}_d(S).$$

$\mathcal{B}(S)$ can be identified with a subalgebra of $\mathbb{R}[x, t]$, where t is a single new variable, by identifying $p \in \mathcal{B}_d(S)$ with $p \cdot t^d$. Also note that

$$\mathcal{B}_d(S_1 \cup S_2) = \mathcal{B}_d(S_1) \cap \mathcal{B}_d(S_2) \quad \text{and} \quad \mathcal{B}(S_1 \cup S_2) = \mathcal{B}(S_1) \cap \mathcal{B}(S_2)$$

holds for $S_1, S_2 \subseteq \mathbb{R}^n$. This follows from the fact that q in the definition of $\mathcal{B}_d(S)$ can always be assumed to be globally non-negative. In fact, one can always take some $C + D\|x\|^{2d}$. Finally note that $\mathcal{B}_d(S)$ and thus $\mathcal{B}(S)$ do only depend on the behaviour of S at infinity; if S is changed inside of a compact set, no changes in $\mathcal{B}_d(S)$ and $\mathcal{B}(S)$ occur. In this paper, we consider the following questions:

Question 1.1. Is $\mathcal{B}(S)$ a finitely generated algebra?

Question 1.2. Is $\mathcal{B}_0(S)$ a finitely generated algebra?

Question 1.3. Is every $\mathcal{B}_d(S)$ a finitely generated $\mathcal{B}_0(S)$-module?

Question 1.4. If $\mathcal{B}_0(S) = \mathbb{R}$, is every $\mathcal{B}_d(S)$ a finite dimensional vector space?