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Flat endofinite modules, prime ideals, and duality

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Bijective correspondences are established between endofinite injective left modules, endofinite flat right modules, finite collections of minimal noetherian prime ideals, normalized rank functions on left ideals and characters. Endofinite flat modules are identified as flat covers of modules associated to a minimal noetherian prime ideal, while endofinite flat injectives are characterized by localizations with a semiprimary QF-3 quotient ring.

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Introduction

Endofinite modules were introduced and first studied by Crawley-Boevey [14] in the context of finite dimensional algebras A over a field. If indecomposable and infinite-dimensional, they are called *generic modules* because of their intimate relationship to infinite series of finite dimensional indecomposables. Every finite dimensional A-module is endofinite, but the inclusion of infinite-dimensional modules led to a new concept of tameness and advanced the geometric understanding of representation categories. H. Krause [26] extended the genericity concept to arbitrary module categories, defining an indecomposable module to be *generic* if it is length-finite over its endomorphism ring but not finitely presented. The most basic example is given by a polynomial ring $K[x]$ over a field K, where the rational function field $K(x)$ is the unique generic $K[x]$-module and every indecomposable finitely presented $K[x]$-module occurs as a subfactor of $K(x)$. In this archetypical example, the generic $K[x]$-module is flat and injective.

To motivate the idea behind our study, it is useful to view module theory in a slightly more general framework. Let $\mathscr{C}$ be an additive category, and let $\mathbf{Mod}(\mathscr{C})$ denote the category of *$\mathscr{C}$-modules*, that is, additive functors from $\mathscr{C}^{\mathrm{op}}$ to the category $\mathbf{Mod}(\mathbb{Z})$ of abelian groups. If $\mathscr{C}$ is the category $\mathbf{add}(R)$ of finitely generated projective (left) modules over a ring R, the category $\mathbf{Mod}(\mathscr{C})$ can be identified with $\mathbf{Mod}(R)$, the category of R-modules. By $\mathbf{mod}(R)$ we denote the category of finitely presented R-modules. Then $\mathbf{com}(R) := \mathbf{mod}(R^{\mathrm{op}})^{\mathrm{op}}$ is the *right abelian* [33] mirror image of $\mathbf{mod}(R)$: The category $\mathbf{add}(R)$ of projective objects in $\mathbf{mod}(R)$ is equivalent to the category of injective objects in $\mathbf{com}(R)$. Note that every object M of an additive category with splitting idempotents – a "variety" in the sense of M. Auslander – generates a full subcategory equivalent to $\mathbf{add}(R)$ with $R := \mathrm{End}(M)^{\mathrm{op}}$. Thus, in a non-commutative sense, the categories $\mathbf{add}(R)$ can be regarded as "affine" pieces of the variety.

In this paper, we focus upon injective versus flat endofinite modules. As every endofinite module over a ring R is injective in the category of modules over $\mathbf{com}(R)$, studying injective endofinite modules means to replace $\mathbf{com}(R)$ by $\mathbf{add}(R)$. Except that $\mathbf{add}(R)$ is affine in the above sense, the descent from $\mathbf{com}(R)$ to $\mathbf{add}(R)$ leads to an even more general situation: As a right abelian category, $\mathbf{com}(R)$ admits kernels and cokernels, while both need not exist in $\mathbf{add}(R)$. As expected, the home advantage of studying endofinite modules in a more familiar context, one step "more inside" $\mathbf{Mod}(R)$, has led to new insights, including precise connections to Gabriel localization, minimal prime ideals, rank functions on left ideals and characters on the module category – ready to be evaluated for arbitrary endofinite modules.

A first challenge for this program has been the search for a proper specialization of Crawley-Boevey's and Herzog's duality [15,19] between left and right endofinite modules to the framework of endofinite injectives in $\mathbf{Mod}(R)$. It was not clear in advance if such a duality survives in our case. Inspired by the left–right symmetry of Schofield's Sylvester module rank functions [36], Crawley-Boevey obtained duality via characters on $\mathbf{com}(R)$. For noetherian rings R, characters were studied independently as *additive*