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Deformations of modules of maximal grade and the Hilbert scheme at determinantal schemes

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ABSTRACT

Let R be a polynomial ring and M a finitely generated graded R-module of maximal grade (which means that the ideal $I_t(\mathcal{A})$ generated by the maximal minors of a homogeneous presentation matrix, $\mathcal{A}$, of M has maximal codimension in R). Suppose $X := \operatorname{Proj}(R/I_t(\mathcal{A}))$ is smooth in a sufficiently large open subset and $\dim X \geqslant 1$. Then we prove that the local graded deformation functor of M is isomorphic to the local Hilbert (scheme) functor at $X \subset \operatorname{Proj}(R)$ under a weak assumption which holds if $\dim X \geqslant 2$. Under this assumption we get that the Hilbert scheme is smooth at (X), and we give an explicit formula for the dimension of its local ring. As a corollary we prove a conjecture of R.M. Miró-Roig and the author that the closure of the locus of standard determinantal schemes with fixed degrees of the entries in a presentation matrix is a generically smooth component V of the Hilbert scheme. Also their conjecture on the dimension of V is proved for $\dim X \geqslant 1$. The cohomology $H^i_*(\mathcal{N}_X)$ of the normal sheaf of X in $\operatorname{Proj}(R)$ is shown to vanish for $1 \leqslant i \leqslant \dim X - 2$. Finally the mentioned results, slightly adapted, remain true replacing R by any Cohen–Macaulay quotient of a polynomial ring.

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1. Introduction

Determinantal objects are central in many areas of mathematics. In algebraic geometry determinantal schemes defined by the vanishing of the $p \times p$-minors of a homogeneous polynomial matrix, may be used to describe classical schemes such as rational normal scrolls and other fibered schemes, Veronese and Segre varieties and Secant schemes to rational normal curves and Segre varieties [20,3]. Throughout the years many nice properties are detected for determinantal schemes, e.g. they are arithmetically Cohen–Macaulay with rather well understood free resolutions and singular loci, see [11,12,37,50], and see [6,5,13,16,30,36,40] for history and other important contributions.

In this paper we study the Hilbert scheme along the locus of determinantal schemes. More precisely we study deformations of modules of maximal grade over a polynomial ring R and establish a very strong connection to corresponding deformations of determinantal schemes in $\mathbb{P}^n$. Recall that the grade g of a finitely generated graded R-module M is the grade of its annihilator $I := \operatorname{ann}(M)$, i.e. $g = \operatorname{depth}_I R = \dim R - \dim R/I$. We say a scheme $X \subset \mathbb{P}^n$ of codimension c is *standard determinantal* if its homogeneous saturated ideal is equal to the ideal $I_t(\mathcal{A})$ generated by the $t \times t$ minors of some homogeneous $t \times (t+c-1)$ matrix $\mathcal{A} = (f_{ij})$, $f_{ij} \in R$. If M is the cokernel of the map determined by $\mathcal{A}$, then $g = c$ because the radicals of I and $I_t(\mathcal{A})$ are equal. Moreover M has maximal grade if and only if $X = \operatorname{Proj}(A)$, $A := R/I_t(\mathcal{A})$ is standard determinantal. In this case $\operatorname{ann}(M) = I_t(\mathcal{A})$ for $c \geqslant 2$ by [7].

Let $\operatorname{Hilb}^p(\mathbb{P}^n)$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^n$ of dimension $n-c \geqslant 0$ and with Hilbert polynomial p. Given integers $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{t+c-2}$ and $b_1 \leqslant \cdots \leqslant b_t$, $t \geqslant 2$, $c \geqslant 2$, we denote by $W_s(\underline{b};\underline{a}) \subset \operatorname{Hilb}^p(\mathbb{P}^n)$ the stratum of standard determinantal schemes where f_{ij} are homogeneous polynomials of degrees $a_j - b_i$. Inside $W_s(\underline{b};\underline{a})$ we have the open subset $W(\underline{b};\underline{a})$ of determinantal schemes that are generically a complete intersection. The elements are called *good* determinantal schemes. Note that $W_s(\underline{b};\underline{a})$ is irreducible, and $W(\underline{b};\underline{a}) \neq \emptyset$ if we suppose $a_{i-1} - b_i > 0$ for $i \geqslant 1$, see (2.2).

In this paper we determine the dimension of a non-empty $W(\underline{b};\underline{a})$ provided $a_{i-2} - b_i \geqslant 0$ for $i \geqslant 2$ and $n - c \geqslant 1$ (Theorem 5.5, Corollary 5.6). Indeed

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c, \tag{1.1}$$

where λ_c and K_i are a large sum of binomials only involving a_j and b_i (see Conjecture 2.2 and (2.10) for the definition of λ_c and K_i). In terms of the Hilbert function, $H_M(-)$, of M, we may alternatively write (1.1) in the form

$$\dim W(\underline{b};\underline{a}) = \sum_{j=0}^{t+c-2} H_M(a_j) - \sum_{i=1}^{t} H_M(b_i) + 1$$

(Remark 3.9). Moreover we prove that the closure $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of the Hilbert scheme $\operatorname{Hilb}^p(\mathbb{P}^n)$ provided ${}_0\operatorname{Ext}^2_A(M,M)$, the degree