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Growth of cross-characteristic representations of finite quasisimple groups of Lie type

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ABSTRACT

In this paper we give a bound to the number of conjugacy classes of maximal subgroups of any almost simple group whose socle is a classical group of Lie type. The bound is $2n^{5.2} + n \log_2 \log_2 q$, where n is the dimension of the classical socle and q is the size of the defining field. To obtain the bound, we first bound the number of projective cross-characteristic representations of simple groups of Lie type as a function of the representation degree. These bounds are computed for different families of groups separately. In the computation, we use information on conjugacy class numbers, minimal character degrees and gaps between character degrees.

1. Introduction

Let H be a finite quasisimple group, with $H/Z(H)$ a simple group of Lie type defined over a field of characteristic p. We are interested in the number of inequivalent n-dimensional irreducible modular representations of H. In [1], R. Guralnick, M. Larsen and P.H. Tiep obtain an upper bound of $n^{3.8}$ for the number of irreducible representations

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with dimension at most n in the defining characteristic p. They use their result to find an asymptotic bound for the number $m(G)$ of conjugacy classes of maximal subgroups of an almost simple group G with socle a group of Lie type. This bound is given as

$$m(G) < ar^6 + br \log\log q,$$

where a and b are unknown constants, and r and q are the rank of the socle and the size of its defining field, respectively.

In this paper, we sharpen the mentioned result in the case where the socle is a classical group, as follows:

Theorem 1.1. *Assume G is a finite almost simple group with socle a classical group of dimension n over the field $\mathbb{F}_q$. Let $m(G)$ denote the number of conjugacy classes of maximal subgroups of G not containing the socle. Then*

$$m(G) < 2n^{5.2} + n \log_2 \log_2 q.$$

Note that there are no unknown constants left in Theorem 1.1. To prove the result, we need to study the representation growth of groups of Lie type over fields of characteristic different from p. Define $r_n(H, \ell)$ as the number of inequivalent irreducible n-dimensional representations of H over the algebraic closure of the finite prime field of characteristic $\ell \neq p$. Also, write $r_n^f(H, \ell)$ for the number of such said representations that are in addition faithful. If $\mathcal{L}$ is a family of finite quasisimple groups of Lie type, we denote

$$s_n(\mathcal{L}, \ell) = \sum_{H \in \mathcal{L}} r_n^f(H, \ell).$$

The sum is taken over a set of representatives H of isomorphism classes of quasisimple groups belonging to $\mathcal{L}$. We will present upper bounds for the growth of $s_n(\mathcal{L}, \ell)$ for different family of groups of Lie type. The upper bounds will have no dependence on ℓ.

Regarding classical groups, we concern ourselves with the following families of quasisimple groups:

- A_1 linear groups in dimension 2 (but see below)
- A' linear groups in dimension at least 3
- ${}^2A$ unitary groups in dimension at least 3
- B orthogonal groups in odd dimension $\geqslant 7$ over a field of odd size
- C symplectic groups in dimension at least 4
- D orthogonal groups of plus type in even dimension $\geqslant 8$
- ${}^2D$ orthogonal groups of minus type in even dimension $\geqslant 8$.

From family A_1, we also exclude the linear groups $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$ and $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$, as well as all their covering groups.