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Dimensions of triangulated categories with respect to subcategories ☆

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ABSTRACT

This paper introduces a concept of dimension of a triangulated category with respect to a fixed full subcategory. For the bounded derived category of an abelian category, upper bounds of the dimension with respect to a contravariantly finite subcategory and a resolving subcategory are given. Our methods not only recover some known results on the dimensions of derived categories in the sense of Rouquier, but also apply to various commutative and non-commutative noetherian rings.

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1. Introduction

The notion of dimension of a triangulated category was introduced by Rouquier [32] based on work of Bondal and van den Bergh [15] on Brown representability. It measures how many extensions are needed to build the triangulated category out of a single object, up to finite direct sum, direct summand and shift. For results on the dimensions of triangulated categories, we refer to [1,14,18,27,36] for instance.

It is still a hard problem in general to give a precise value of the dimension of a given triangulated category. The aim of this paper is to provide new information on this problem. We give upper bounds of the dimensions of derived categories in terms of global dimensions. A prototype of our approach is given by the inequality

$$\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \leqslant \operatorname{gl.dim} \operatorname{End}_{\Lambda}(M) \tag{1.0.1}$$

for a noetherian algebra Λ and a generator M of Λ [27, (3.4)], where $\operatorname{tri.dim} \mathcal{T}$ denotes the dimension of a triangulated category $\mathcal{T}$. This observation was applied to study representation dimension [23,27,29,31].

Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a full subcategory. This paper introduces and studies *the dimension*

$$\mathcal{X}\text{-}\operatorname{tri.dim} \mathcal{T}$$

of $\mathcal{T}$ *with respect to* $\mathcal{X}$, which measures how many extensions are needed to build $\mathcal{T}$ out of $\mathcal{X}$, up to finite direct sum, direct summand and shift. A similar notion called *level* was studied by Avramov, Buchweitz, Iyengar and Miller [12]; it is defined for each object of $\mathcal{T}$.

In this paper, first we generalize the inequality (1.0.1) by replacing the right hand side with the global dimension of a functor category. In representation theory, to study a category $\mathcal{X}$ of representations of Λ such as $\operatorname{mod} \Lambda$, $\mathsf{CM}(\Lambda)$ and $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$, the functor category $\operatorname{mod} \mathcal{X}$ plays a crucial role [9]. For example, it follows from a basic observation in Auslander–Reiten theory that projective resolutions of simple objects in $\operatorname{mod} \mathcal{X}$ correspond to almost split sequences in $\mathcal{X}$ [4,35]. Our first main result is the following theorem.

Theorem 1.1. *Let $\mathcal{A}$ be an abelian category, and $\mathcal{X}$ a contravariantly finite subcategory that generates $\mathcal{A}$. Then*

$$\mathcal{X}\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leqslant \operatorname{gl.dim}(\operatorname{mod} \mathcal{X}).$$

One can recover (1.0.1) by letting $\mathcal{X} = \operatorname{add} M$. For a cotilting module T we apply this result to the full subcategory $\mathcal{X}_T$ of $\operatorname{mod} \Lambda$ consisting of modules X with $\operatorname{Ext}_{\Lambda}^{>0}(X, T) = 0$ to get:

Corollary 1.2. *Let Λ be a noetherian ring and T a cotilting module. Then it holds that $\mathcal{X}_T\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \leqslant \max\{1, \operatorname{inj.dim} T\}$.*

For example, when R is a Cohen–Macaulay local ring with a canonical module, the above result induces an inequality

$$\mathsf{CM}(R)\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leqslant \max\{1, \dim R\}. \tag{1.2.1}$$

In particular, if R has finite Cohen–Macaulay representation type, then the inequality $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leqslant \max\{1, \dim R\}$ holds.

Next, we give another approach based on Cartan–Eilenberg resolutions in derived categories. We prove the following theorem as our second main result.