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Normal sections, class sizes and solvability of finite groups

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ABSTRACT

If G is a finite group, we show that any normal subgroup of G which has exactly three G-conjugacy class sizes is solvable. Thus, we give an extension for normal subgroups of the classical N. Itô's theorem which asserts that those finite groups having three class sizes are solvable, and particularly, a new proof of it is provided. In order to do this, we investigate the structure of a normal section N/K of G such that every element in N lying outside of K has the same G-class size.

1. Introduction

Let G be a finite group and N be a normal subgroup of G. Recent research works have shown that the set of sizes of those conjugacy classes of G contained in N, also called G-class sizes of N and denoted by $\mathrm{cs}_G(N)$, exerts a strong influence on the structure of N. In general, from the set $\mathrm{cs}_G(N)$, very little information on the class sizes of N can be obtained. Nevertheless, it is surprising that the G-class sizes of normal subgroups still seem to keep control on their structure. This has emerged as a new useful technique to obtain information regarding normal subgroups, and moreover, this approach has the advantage of enabling to argue by induction on the order of N.

In [3] it is proved that every normal subgroup of G having two G-class sizes is nilpotent, and in fact, such subgroup is either abelian or the direct product of a p-group for a prime p and a central subgroup of G. This result obviously extends the celebrated N. Itô's theorem on the nilpotency of

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groups with exactly two class sizes. However, while Itô's proof is quite elementary, the proof of the above extension is much deeper.

The solvability of groups with three conjugacy class sizes is a more complex problem. N. Itô proved in [14] that such groups are solvable by appealing to Feit–Thompson's theorem and some deep classification theorems by M. Suzuki. The result was simplified years later by J. Rebmann [15] for F-groups (those groups which do not have any two centralizers of noncentral elements such that one properly contains the other). Afterwards, A.R. Camina showed in [6] by using different techniques from those employed in [15], that if G is not an F-group and has three class sizes, then G is the direct product of an abelian subgroup and a subgroup whose order involves no more than two primes. In 2009, S. Dolfi and E. Jabara completely determined their structure and their proof was based on the solvability of this kind of groups [7].

In [1], we obtained the solvability and the structure of a normal subgroup N of G with three G-class sizes just when $\mathrm{cs}_G(N) = \{1, m, n\}$ and m does not divide n. The proof of this result sets up a generalization and a subsequent classification of the concept of F-group for normal subgroups. In [4] the authors asked whether every normal subgroup having three G-class sizes should be solvable or not and thus, the current open problem is reduced to the case of a normal subgroup N with $\mathrm{cs}_G(N) = \{1, m, n\}$ such that m does divide n. In this latter paper, the solvability of N is proved in a particular case and distinct techniques from those in [1] are required. For instance, Theorem B of [4] (see Theorem 6) plays a crucial role so as to get the solvability in the general case.

In order to prove our main goal, Theorem A, we will use the structure of "CP-groups", that is, the groups in which every element has prime power order. These groups also appear in a natural way in the main results of [3] and [4].

Theorem A. *If N is a normal subgroup of a finite group G and $|\mathrm{cs}_G(N)| = 3$, then N is solvable.*

Recently [2], it has been determined the solvability and the structure of those normal subgroups of G having at most two p-regular G-class sizes (a p-regular class is a class of an element whose order is not divisible by p), where p is a prime number. This turns out to be a key result to progress in the proof of Theorem A. We would like to remark that in all the cited papers on G-classes [1–4] the Classification of the Finite Simple Groups is required.

Another central tool is our analysis of the structure of normal sections in a group G involving certain hypotheses on the G-class sizes. In Theorem B, which has interest on its own, we put forward an extension of an M.I. Isaacs' result (see [11]) about groups having a proper normal subgroup N such that all of the conjugacy classes of G lying outside of N have equal size. Extending Isaacs' definition, we give the following

Definition. We say that a normal section N/K of a group G, that is, $K, N \trianglelefteq G$ with $K \subseteq N$, satisfies condition $(*)$ over G when N is a nonabelian normal subgroup of G such that all the G-conjugacy classes in N lying outside of K have equal size.

Theorem B. *Let N/K be a normal section satisfying $(*)$ over G.*

(i) *If $\mathbf{Z}(N) \nsubseteq K$, then N/K is a p-group for some prime p and N/K is either abelian or has exponent p.*
(ii) *If $\mathbf{Z}(N) \subseteq K$, then either N/K is cyclic or is a* CP-*group. If N/K is not a* CP-*group, then N has abelian Hall π-subgroups and a normal π-complement, where π is the set of prime divisors of $|N/K|$.*

If x is any element of a group G, we denote by x^G the conjugacy class of x in G and by $|x^G|$ the G-conjugacy class size of x. This is also called the index of x in G. If p is a prime and n is an integer, we use n_p to denote the p-part of n, and $\pi(n)$ is the set of primes dividing n, and $\pi(G)$ is $\pi(|G|)$. On the other hand, if H is a group and q a prime, then we will denote by H_q a Sylow q-subgroup of H, and analogously for a set of primes π, H_π denotes a Hall π-subgroup of H. For the rest of the notation, we will follow [12].