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Deformations of Jordan algebras of dimension four

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ABSTRACT

We study the variety $\mathcal{J}or_4$ of four-dimensional Jordan algebras on $\mathbf{k}^4$, for $\mathbf{k}$ an algebraically closed field of characteristic $\neq 2$. We describe its irreducible components and prove that $\mathcal{J}or_4$ is the union of Zariski closures of the orbits of 10 rigid algebras.

1. Introduction

The goal of this paper is to study the variety of Jordan algebras of dimension four over an algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$ and to describe its irreducible components. Namely, let V be a $\mathbf{k}$-vector space of finite dimension n, then the bilinear maps $\mathrm{Hom}_{\mathbf{k}}(\mathrm{V} \times \mathrm{V}, \mathrm{V}) = \mathrm{V}^* \otimes \mathrm{V}^* \otimes \mathrm{V}$ form a vector space of dimension n^3 and it has the structure of an affine variety $\mathbf{k}^{n^3}$. The algebras satisfying the Jordan identity form a Zariski-closed affine subset of $\mathbf{k}^{n^3}$ which we will call the variety of Jordan algebras of dimension n, $\mathcal{J}or_n$.

The problem of finding procedure for determining generic points of $\mathcal{J}or_n$ or equivalently irreducible components of the algebraic variety, could be formulated geometrically as follows: the linear group $\mathrm{G} = \mathrm{GL}(\mathrm{V})$ operates on $\mathcal{J}or_n$ by conjugation, decomposing $\mathcal{J}or_n$ into G-orbits which correspond to the classes of isomorphic Jordan algebras. If an algebra $\mathcal{J}$ lies in the Zariski closure of the orbit

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of a (non-isomorphic) algebra $\mathcal{J}'$ in the variety, then we will say that $\mathcal{J}'$ is a deformation of $\mathcal{J}$. An algebra $\mathcal{J}$ whose orbit $\mathcal{J}^{\mathrm{G}}$ is Zariski-open in $\mathcal{J}\mathrm{or}_n$ is called rigid. They are of particular interest since the Zariski closure of the orbits of rigid algebra gives an irreducible component of the variety.

Note that analogously one defines the varieties of Lie and associative algebras. Denote them Lie_n and Assoc_n respectively. The geometry of both is rather complicated. In 1890 E. Study in [1] considered complex associative algebras of dimension four and showed that it is impossible to find the generic algebra, or equivalently, that Assoc_4 has more than one irreducible component.

In 1964, M. Gerstenhaber in his work [2] turned the geometric definition of deformation into analytical one, namely he introduced the notion of formal (infinitesimal) deformation between associative algebras. Let A be an n-dimensional associative algebra and consider an n^3-tuple $g = \{g_{ijk}(t)\}$ of power series in the variable t, such that $g(t)$ defines the associative multiplication on $A_t = \mathrm{V} \otimes \mathbf{k}(t)$ and $g_{ijk}(0)$ coincide with the structure constants of A then we say that A has been deformed into A_t. One can check that the analytical definition implies the geometric one. In particular an algebra A is rigid if any $g_{ijk}(t)$ satisfying the above conditions defines the algebra A_t isomorphic to A for every $t \in \mathbf{k}$. Also, it was shown that if the second cohomology group of $A \in \mathrm{Assoc}_n$ with coefficients in A, $H^2(A, A)$, is trivial then A is a rigid algebra. Four years later, the analytical deformation theory introduced by Gerstenhaber for associative algebras was extended to Lie algebras by Nijenhuis and Richardson in [3].

There were two further papers of F. Flanigan, first [4, 1968], where he compared the structure of the deformed algebra A_t with the structure of original algebra A, see Fact 8 for the analogous theorem for Jordan algebras. Also in [5, 1969] he showed that the irreducible component of Assoc_n is either the closure of the orbit of a rigid algebra or the closure of the union of the orbits of non-isomorphic algebras, called semi-rigid there.

In [6, 1974] P. Gabriel described all generic algebras for the closed subvariety of unital associative algebras of Assoc_n for $n \leqslant 4$. Also this is a good reference to check the basic characteristics which are almost constant on the irreducible components such as the dimension of the radical of algebras, the dimension of the automorphism group, etc. Further in [7, 1979], Gabriel's student G. Mazzola presented the inclusion diagram of the orbits of five-dimensional unital associative algebras and proved that there are ten irreducible components (generic algebras) in this subvariety of Assoc_5. The other questions considered by Mazzola dealt with the variety of unital commutative algebras inside of Assoc_n, see [8, 1980] and the references therein. Seven years later, in [9] were described the components of Lie_n for $n \leqslant 6$. There are further recent developments in this theory when the variety of representations of algebras is considered instead of the variety of algebras.

As to the variety $\mathcal{J}\mathrm{or}_n$ the references are rather recent. In [10, 2005] all deformations in $\mathcal{J}\mathrm{or}_3$ were described, in [11, 2006] the closed subvarieties of unital Jordan algebras of $\mathcal{J}\mathrm{or}_n$, $n \leqslant 5$, were considered, the irreducible components being described. Also, some properties and facts known for Assoc_n and Lie_n were extended to the case of Jordan algebras. Finally, in [12, 2011] infinitesimal deformations were used to study nilpotent Jordan algebras of dimension four. The goal of this paper is to generalize the results of [11] and [12] and to obtain a complete description of the irreducible components of $\mathcal{J}\mathrm{or}_4$.

For the standard terminology on Jordan algebras, the reader is referred to the books of R. Schafer [13] and N. Jacobson [14], for concepts from deformation theory see [2]. In the following, we will work over an algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$ and, furthermore, all Jordan algebras are assumed to be of finite dimension over $\mathbf{k}$.

The paper is organized as follows. Section 2 contains some preliminaries on Jordan algebras, also we provide the list of Jordan algebras of dimension less or equal to four over an algebraically closed field $\mathbf{k}$. In Section 3, we define the variety of Jordan algebras $\mathcal{J}\mathrm{or}_n$ and write down its basic properties and useful characteristics of $\mathcal{J}\mathrm{or}_n$, while in Section 4, we apply them to establish the list of G-orbits constructing deformations between the algebras in $\mathcal{J}\mathrm{or}_n$ for $n \leqslant 4$.

2. Jordan algebras of small dimensions

In this section we present the basic concepts of Jordan algebras as well as we provide the list of Jordan algebras of dimension less than or equal to four.