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Commutator theory for loops ☆

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ABSTRACT

Using the Freese–McKenzie commutator theory for congruence modular varieties as the starting point, we develop commutator theory for the variety of loops. The fundamental theorem of congruence commutators for loops relates generators of the congruence commutator to generators of the total inner mapping group. We specialize the fundamental theorem into several varieties of loops, and also discuss the commutator of two normal subloops.

Consequently, we argue that some standard definitions of loop theory, such as elementwise commutators and associators, should be revised and linked more closely to inner mappings. Using the new definitions, we prove several natural properties of loops that could not be so elegantly stated with the standard definitions of loop theory. For instance, we show that the subloop generated by the new associators defined here is automatically normal. We conclude with a preliminary discussion of abelianess and solvability in loops.

1. Introduction

The two primary influences on modern loop theory come from group theory and universal algebra, a fact that is reflected already in the definition of a loop. Using the group-theoretical approach, a *loop* is a nonempty set Q with identity element 1 and with binary operation $\cdot$ such that for every $a, b \in Q$ the equations $a \cdot x = b$, $y \cdot a = b$ have unique solutions $x, y \in Q$. The implied presence of divisions is

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made explicit in the equivalent universal algebraic definition due to Evans [11]: a *loop* is a universal algebra $(Q, 1, \cdot, \backslash, /)$ satisfying the identities

$$x \cdot 1 = x = 1 \cdot x, \qquad x\backslash(x \cdot y) = y, \qquad x \cdot (x\backslash y) = y, \qquad (y \cdot x)/x = y, \qquad (y/x) \cdot x = y.$$

It is not difficult to see that associative loops are precisely groups, where we write $x^{-1}y$ and xy^{-1} in place of $x\backslash y$ and x/y, respectively.

The most influential text on loop theory in the English-speaking world is the book of Bruck [5]. Although its title "A Survey of Binary Systems" and its opening chapters are rather encompassing, it focuses on and culminates in the study of Moufang loops, a variety of loops with properties close to groups. It is therefore natural that Bruck's definitions are rooted mostly in group theory. For instance, a subloop N of a loop Q is said to be *normal* in Q if

$$xN = Nx, \qquad x(yN) = (xy)N, \qquad N(xy) = (Nx)y$$

for every $x, y \in Q$, the *center* $Z(Q)$ of Q is defined as

$$Z(Q) = \bigl\{a \in Q;\ ax = xa,\ a(xy) = (ax)y,\ x(ay) = (xa)y,\ x(ya) = (xy)a \text{ for every } x, y \in Q\bigr\},$$

and the elementwise *commutator* $[x, y]$ and *associator* $[x, y, z]$ are defined as the unique solutions to the equations

$$xy = (yx)[x, y], \qquad (xy)z = \bigl(x(yz)\bigr)[x, y, z],$$

respectively. The *associator subloop* $A(Q)$ of Q is the smallest normal subloop of Q such that $Q/A(Q)$ is a group, or, equivalently, the smallest normal subloop of Q containing all associators $[x, y, z]$ of Q. The *derived subloop* Q' of Q is the smallest normal subloop of Q such that Q/Q' is an abelian group, or, equivalently, the smallest normal subloop of Q containing all commutators $[x, y]$ and all associators $[x, y, z]$ of Q. With $Q^{[0]} = Q$, $Q^{[i+1]} = (Q^{[i]})'$, a loop is called *solvable* if $Q^{[n]} = 1$ for some n.

It is easy to see that in groups the above concepts specialize to the usual group-theoretical notions (the associator and the associator subloop being void). Bruck's deep results [5] showed that the group-theoretical definitions are sensible in Moufang loops, as did Glauberman's extension of the Feit–Thompson Odd Order Theorem to Moufang loops [17].

As it turns out, the normality, the center and the derived notion of central nilpotency are the correct concepts for loops even from the universal algebraic point of view. (For normality this was known already to Bruck. For the center and central nilpotency this is probably folklore, but since we were not able to find a proof in the literature, we present it at the end of this paper for the convenience of the reader.) It is therefore not surprising that central nilpotency played a prominent role in the development of loop theory, as witnessed by the 42 papers listed in MathSciNet under primary classification 20N05 and with one of the words "nilpotent", "nilpotency" or "nilpotence" in the title. We mention [8,9,17,18,21,25,29,30,32,33,37] as a representative sample.

But the commutators and associators did not fare as well, and neither did the concept of solvability. The inadequacies of the elementwise associators were first pointed out by Leong [24]; see below for more details. There is no established notion of commutator of two normal subloops and, in contrast to nilpotency, there are only 9 papers on MathSciNet under 20N05 and with one of the words "solvable", "solvability", "soluble" or "solubility" in the title.

We maintain that this is not a coincidence, but rather a consequence of the fact that the elementwise associators, the commutator theory and solvability were not well conceived in loop theory. This is somewhat surprising, since loops are known to be congruence modular (they possess a Mal'tsev term), the general commutator theory for congruence modular varieties [12] has been developed more than 25 years ago, and, furthermore, the original impetus for the commutator theory came from