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# The fundamental group of affine curves in positive characteristic

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ABSTRACT

It is shown that the commutator subgroup of the fundamental group of a smooth irreducible affine curve over an uncountable algebraically closed field $k$ of positive characteristic is a profinite free group of rank equal to the cardinality of $k$.



## 1. Introduction

The algebraic (étale) fundamental group of an affine curve over an algebraically closed field $k$ of positive characteristic has a complicated structure. It is an infinitely generated profinite group, in fact the rank of this group is same as the cardinality of $k$. The situation when $k$ is of characteristic zero is simpler to understand. The fundamental group of a smooth curve over an algebraically closed field of characteristic zero is just the profinite completion of the topological fundamental group [17, XIII, Corollary 2.12, page 392]. In positive characteristic as well, Grothendieck gave a description of the prime-to-$p$ quotient of the fundamental group of a smooth curve which is in fact analogous to the characteristic zero case. From now on, we shall assume that the characteristic of the base field $k$ is $p > 0$. Consider the following exact sequence for the fundamental group of a smooth affine curve $C$.

$$1 \to \pi_1^c(C) \to \pi_1(C) \to \pi_1^{ab}(C) \to 1$$

where $\pi_1^c(C)$ and $\pi_1^{ab}(C)$ are the commutator subgroup and the abelianization of the fundamental group $\pi_1(C)$ of $C$ respectively. In [11], a description of $\pi_1^{ab}(C)$ was given [11, Corollary 3.5] and it was

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also shown that $\pi_1^c(C)$ is a free profinite group of countable rank if $k$ is countable [11, Theorem 1.2]. In fact some more exact sequences with free profinite kernel like the above were also observed [11, Theorem 7.1]. Later using somewhat similar ideas and some profinite group theory Pacheco, Stevenson and Zalesskii claim to find a condition for a closed normal subgroup of $\pi_1(C)$ to be profinite free of countable rank [12] but unfortunately there seems to be a gap in their argument as Example 3.16 suggests.

A consequence of the main result of this paper generalizes [11, Theorem 1.2] to uncountable fields.

**Theorem 1.1.** *Let $C$ be a smooth affine curve over an algebraically closed field $k$ (possibly uncountable) of characteristic $p$ then $\pi_1^c(C)$ is a free profinite group of rank* $\mathrm{card}(k)$.

This answers a question of Harbater and Zalesskii who had asked the author in an email communication if the above is true or at least whether there exist a closed subgroup of $\pi_1(C)$ which is free of rank same as $\mathrm{card}(k)$. Let $P_g(C)$ be the intersection of all index $p$ normal subgroups of $\pi_1(C)$ corresponding to étale covers of $C$ of genus at least $g$ (see Definition 3.4). In the main theorem (Theorem 3.6) it is shown that if $\Pi$ is a closed normal subgroup of $\pi_1(C)$ of rank $\mathrm{card}(k)$ such that $\pi_1(C)/\Pi$ is abelian, $\Pi \subset P_g(C)$ for some $g \geqslant 0$ and for every finite simple group $S$ there exist a surjection from $\Pi$ to $\mathrm{card}(k)$ copies of $S$ then $\Pi$ is profinite free.

As a consequence we get the following result.

**Corollary 1.2.** *Let $\Pi$ be a closed normal subgroup of $P_g(C)$ for some $g \geqslant 0$. If rank of $P_g(C)/\Pi$ is strictly less than* $\mathrm{card}(k)$ *then $\Pi$ is profinite free of rank* $\mathrm{card}(k)$.

**Proof.** Note that $P_g(C)$ is a profinite free group of rank $\mathrm{card}(k)$ by Corollary 3.15. So the corollary follows from Melnikov's result on freeness of a normal subgroup of a profinite free group [16, Theorem 8.9.4]. □

The existence of wildly ramified covers is the primary reason why so little is known about fundamental groups of affine curves. More precisely, there is a positive dimensional configuration space of $p$-cyclic Artin–Schreier covers of the affine line (see [14]), and while this family is relatively well understood, the structure of $\pi_1(\mathbb{A}^1)$ remains elusive. This is because we do not know how the various wildly ramified covers fit in with the tamely ramified covers in the tower of covers over $\mathbb{A}^1$. This also suggests that the fundamental group of an affine curve in positive characteristic contains much more information about the curve than in characteristic zero case. In fact Harbater and Tamagawa have conjectured that the fundamental group of a smooth affine curve over an algebraically closed field of characteristic $p$ should determine the curve completely (as a scheme) and in particular one should be able to recover the base field. Harbater and Tamagawa have shown some positive results supporting the conjecture. See [10, Section 3.4], [8], [18] and [19] for more details.

The above theorem on the commutator subgroup can also be interpreted as an analogue of the Shafarevich conjecture for global fields. The Shafarevich conjecture says that the commutator subgroup of the absolute Galois group of the rational numbers $\mathbb{Q}$ is a profinite free group of countable rank. David Harbater [6], Florian Pop [13] and later Dan Haran and Moshe Jarden [3] have shown, using different patching methods, that the absolute Galois group of the function field of a curve over an algebraically closed field is profinite free of the rank same as the cardinality of the base field. See [7] for more details on these kind of results and questions.

Though the profinite group structure of the fundamental group of a smooth affine curve is not well understood, a 1994 proof of Abhyankar's 1957 conjecture provides a characterization for a finite group to be a quotient of the fundamental group of a smooth affine curve. For a finite group $G$, let $p(G)$ denote the subgroup of $G$ generated by all the Sylow-$p$ subgroups of $G$. The conjecture was proved by Raynaud for the affine line [15] and by Harbater in general [4].

**Theorem 1.3** *(Harbater, Raynaud). Let $C$ be a smooth affine curve of genus $g$ over an algebraically closed field of characteristic $p$. Let $D$ be the smooth compactification of $C$ and* $\mathrm{card}(D \setminus C) = n + 1$. *Then a finite group $G$ is a quotient of $\pi_1(C)$ if and only if $G/p(G)$ is generated by $2g + n$ elements.*