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Towards connectivity for codimension 2 cycles: Infinitesimal deformations

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ABSTRACT

Let X be a smooth projective variety over an algebraically closed field $k \subset \mathbb{C}$ of characteristic zero, and $Y \subset X$ a smooth complete intersection. The Weak Lefschetz Theorem states that the natural restriction map $\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \to \mathrm{H}^i(Y(\mathbb{C}), \mathbb{Q})$ on singular cohomology is an isomorphism for all $i < \dim(Y)$. The Bloch–Beilinson conjectures on the existence of certain filtrations on Chow groups combined with standard conjectures in the theory of motives imply that a similar result should be true for Chow groups, and, more generally, for motivic cohomology. In this note, we prove a consequence of the *Motivic Weak Lefschetz Conjecture* (see Conjecture 1.2) for codimension 2 cycles.

1. Introduction

In this note we study a motivic analog of the Weak Lefschetz Theorem. Throughout, X will be a smooth projective variety over an algebraically closed field $k \subset \mathbb{C}$ of characteristic zero. Let $\mathrm{H}^*(X) := \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$ denote the singular cohomology with rational coefficients. Let $Y \subset X$ be a smooth complete intersection of dimension n. Then one has the following result:

Theorem 1.1 *(The Weak Lefschetz Theorem). The restriction map $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ is an isomorphism for $i < n$ and injective for $i = n$.*

It is known that étale cohomology, and all other good cohomology theories on smooth projective varieties over k also satisfy the above property. The following conjecture is a motivic analog of the Weak Lefschetz Theorem (see, for instance, [10, 1.5]):

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Conjecture 1.2. *Let X be a smooth projective variety over k and $Y \subset X$ a smooth complete intersection. The natural restriction map $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(Y)_{\mathbb{Q}}$ is an isomorphism for all $p < \dim(Y)/2$.*

The above conjecture can be deduced from more general conjectures of Bloch and Beilinson (see [7]) on the existence of certain filtrations on Chow groups. In fact, one also expects Weak Lefschetz theorems more generally for higher Chow groups and motivic cohomology (cf. Section 2). The case $p = 1$ of the above conjecture is a classical theorem of Lefschetz and Grothendieck (see [4]). In this case, the conjecture even holds integrally.

Theorem 1.3 *(Grothendieck–Lefschetz). Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection such that $\dim(Y) \geqslant 3$. Then the natural restriction map*

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

The Grothendieck–Lefschetz theorem is proved by factoring the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ as a composition

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(Y),$$

where $\mathfrak{X}$ is the formal completion of X along Y, and showing that each of these arrows is an isomorphism. If we adopt a similar strategy to study the higher codimension case, then as a first step, we would like to write the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ as a composition

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(\mathfrak{X}) \to \mathrm{CH}^p(Y).$$

In particular, we would like to define the middle term.

Recall that the Bloch–Quillen formula, which relates the Chow groups of smooth projective varieties to K-cohomology groups, gives a natural isomorphism

$$\mathrm{CH}^p(X) \cong \mathrm{H}^p(X, \mathcal{K}_{p,X}).$$

Here $\mathcal{K}_{p,X}$ is the sheaf associated to the presheaf which sends an open subset $U \subset X$ to $\mathrm{K}_p(U)$, where $\mathrm{K}_p(U)$ is the p-th Quillen K-theory group of the exact category of locally free sheaves on U. Using this formula, we may define the Chow group of codimension p cycles $\mathrm{CH}^p(\mathfrak{X})$ to be $\mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}})$ where $\mathcal{K}_{p,\mathfrak{X}}$ is defined in an analogous manner. One can then show that the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ factors as

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(\mathfrak{X}) \to \mathrm{CH}^p(Y).$$

We refer to Section 3 for details.

There is however, yet another definition for the Chow groups of the formal scheme $\mathfrak{X}$. For any projective system of sheaves $(\mathcal{F}_n)$ on a scheme V, following Jannsen [6], we can consider the *continuous cohomology groups* denoted by $\mathrm{H}^p_{cont}(V, (\mathcal{F}_n))$. The formalism of continuous cohomology, together with the Bloch–Quillen formula then suggests another definition for the Chow group of codimension p cycles of $\mathfrak{X}$, namely $\mathrm{H}^p_{cont}(Y, (\mathcal{K}_{p,Y_n}))$. Here Y_n is the n-th infinitesimal thickening of Y in X such that $Y_0 = Y$. For the purposes of this article, we will work with this definition and set

$$\mathrm{CH}^p_{cont}(\mathfrak{X}) := \mathrm{H}^p_{cont}\big(Y, (\mathcal{K}_{p,Y_n})\big).$$

One can show that the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ factors as a composition (cf. Section 3)

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p_{cont}(\mathfrak{X}) \to \mathrm{CH}^p(Y). \tag{1}$$