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On the Hilbert function of one-dimensional local complete intersections

J. Elias [a,1], M.E. Rossi [b,*,2], G. Valla [b]

[a] Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
[b] Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy

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ABSTRACT

The sequences that occur as Hilbert functions of standard graded algebras A are well understood by Macaulay's theorem; those that occur for graded complete intersections are elementary and were known classically. However, much less is known in the local case, once the dimension of A is greater than zero, or the embedding dimension is three or more.

Using an extension to the power series ring R of Gröbner bases with respect to local degree orderings, we characterize the Hilbert functions H of one-dimensional quadratic complete intersections $A = R/I$, $I = (f, g)$, of type $(2, 2)$ that is, that are quotients of the power series ring R in three variables by a regular sequence f, g whose initial forms are linearly independent and of degree 2. We also give a structure theorem up to analytic isomorphism of A for the minimal system of generators of I, given the Hilbert function.

More generally, when the type of I is $(2, b)$ we are able to give some restrictions on the Hilbert function. In this case we can also prove that the associated graded algebra of A is Cohen–Macaulay if and only if the Hilbert function of A is strictly increasing.

1. Introduction and preliminaries

Let G be a standard graded K-algebra; by this we mean $G = P/I$ where $P = \mathrm{K}[x_1, \dots, x_n]$ is a polynomial ring over the field K and I a homogeneous ideal. It is clear that for every $t \geqslant 0$ the set I_t of

* Corresponding author.
E-mail addresses: elias@ub.edu (J. Elias), rossim@dima.unige.it (M.E. Rossi), valla@dima.unige.it (G. Valla).

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the forms of degree t in P is a K-vector space of finite dimension. For every positive integer t the Hilbert function of G is defined as follows:

$$HF_G(t) = dim_{\mathrm{K}}\, G_t = dim_{\mathrm{K}}\, P_t - dim_{\mathrm{K}}\, I_t = \binom{n+t-1}{t} - dim_{\mathrm{K}}\, I_t.$$

Its generating function $HS_G(\theta) = \sum_{t\in\mathbb{N}} HF_G(t)\theta^t$ is the Hilbert series of G.

The relevance of this notion comes from the fact that in the case I is the defining ideal of a projective variety V, the dimension, the degree and the arithmetic genus of V can be immediately computed from the Hilbert series of P/I.

A fundamental theorem by Macaulay describes exactly those numerical functions which occur as the Hilbert functions of a standard graded K-algebra. Macaulay's theorem says that for each t there is an upper bound for $HF_G(t+1)$ in terms of $HF_G(t)$, and this bound is sharp in the sense that any numerical function satisfying it can be realized as the Hilbert function of a suitable homogeneous standard K-algebra. These numerical functions are called "admissible" and will be described in the next section.

It is not surprising that additional properties yield further constraints on the Hilbert function. Thus, for example, the Hilbert function of a Cohen–Macaulay standard graded algebra is completely described by another theorem of Macaulay which says that the Hilbert series admissible for a Cohen–Macaulay standard graded algebra of dimension d, are of the type

$$\frac{1 + h_1\theta + \cdots + h_s\theta^s}{(1-z)^d}$$

where $1 + h_1\theta + \cdots + h_s\theta^s$ is admissible.

The Hilbert function of a local ring A with maximal ideal $\mathfrak{m}$ and residue field K is defined as follows: for every $t \geqslant 0$

$$HF_A(t) = dim_{\mathrm{K}}\left(\frac{\mathfrak{m}^t}{\mathfrak{m}^{t+1}}\right).$$

It is clear that $HF_A(t)$ is equal to the minimal number of generators of the ideal $\mathfrak{m}^t$ and we can see that the Hilbert function of the local ring A is the Hilbert function of the following standard graded algebra

$$gr_{\mathfrak{m}}(A) = \oplus_{t\geqslant 0}\mathfrak{m}^t/\mathfrak{m}^{t+1}.$$

This algebra is called the associated graded ring of the local ring $(A, \mathfrak{m})$ and corresponds to a relevant geometric construction in the case A is the localization at the origin O of the coordinate ring of an affine variety V passing through O. It turns out that $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *tangent cone* of V at O, which is the cone composed of all lines that are the limiting positions of secant lines to V in O.

Despite the fact that the Hilbert function of a standard graded K-algebra G is so well understood in the case G is Cohen–Macaulay, very little is known in the local case. This is mainly because, in passing from the local ring A to its associated graded ring, many of the properties of A can be lost. This is the reason why we are very far from a description of the admissible Hilbert functions for a Cohen–Macaulay local ring when $gr_{\mathfrak{m}}(A)$ is not Cohen–Macaulay.

An example by Herzog and Waldi (see [12]) shows that the Hilbert function of a one-dimensional Cohen–Macaulay local ring can be decreasing, even the number of generators of the square of the maximal ideal can be less than the number of generators of the maximal ideal itself. Further, without restrictions on the embedding dimension, the Hilbert function of a one-dimensional Cohen–Macaulay local ring can present arbitrarily many "valleys" (see [7]).