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Canonical systems and their limits on stable curves

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ABSTRACT

We propose an object called 'sepcanonical system' on a stable curve X_0 which is to serve as limiting object – distinct from other such limits introduced previously – for the canonical system, as a smooth curve degenerates to X_0. First, for curves which cannot be separated by 2 or fewer nodes (the so-called '2-inseparable' curves), the sepcanonical system consists of the sections of the dualizing sheaf, and fails to be very ample iff X_0 is a limit of smooth hyperelliptic curves (such X_0 are called 2-inseparable hyperelliptics). For general, 2-separable curves X_0, this assertion is false, leading us to introduce the sepcanonical system, which is a collection of linear systems on the '2-inseparable parts' of X_0, each associated to a different twisted limit of the canonical system, where the entire collection varies smoothly with X_0. To define sepcanonical system, we must endow the curve with extra structure called an 'azimuthal structure'. We show (Theorem 6.5) that the sepcanonical system is 'essentially very ample' unless the curve is a tree-like arrangement of 2-inseparable hyperelliptics. In a subsequent paper [11] we will show that the latter property is equivalent to the curve being a limit of smooth hyperelliptics, and will essentially give defining equation for the closure of the locus of smooth hyperelliptic curves in the moduli space of stable curves.

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Introduction

A big part of the geometry of a nonsingular curve revolves around its canonical system and canonical map, which is always an embedding except for the well-understood exception of hyperelliptic curves. The purpose of this paper is to identify and study an appropriate 'limiting object', called *sepcanonical system*, of the canonical system as the curve degenerates to a stable one, an issue that is closely related to the extrinsic geometry, especially defining equations, of the closure of the hyperelliptic locus in the moduli space of stable curves. The main results are Theorem 5.10, which is (equivalent to) a characterization of the limit object by data on the limit curve, and Theorem 6.5 which characterizes limits of smooth hyperelliptic curves in terms of sepcanonical systems.

The default choice for the limiting object is certainly the canonical system itself, i.e. the linear system associated to the dualizing sheaf, on the limiting stable curve. This choice will be analyzed in Sections 1–3 below. What the analysis shows is that the canonical system on a stable curve X_0 is a good choice of limit as long as X_0 is '2-inseparable' in the sense that it cannot be disconnected by removing 2 or fewer nodes. We will show (Theorem 2.13) that a 2-inseparable curve is a limit of smooth hyperelliptic curves iff its dualizing sheaf is not very ample. Those curves were classified long ago by Catanese [5].

Going beyond 2-inseparable stable curves X_0, it becomes less than clear that the canonical system on X_0 is the correct limiting object: for example it is no longer true that X_0 is the limit of hyperelliptics whenever its canonical map is not birational; indeed the locus of stable curves of genus g whose canonical map is not birational has components of fixed codimension c independent of g contained in the boundary (in fact, in the boundary component Δ_0) of the Deligne–Mumford Moduli space $\overline{\mathcal{M}}_g$. One the other hand, the analysis of 2-inseparable curves suggests viewing these as atoms for the purpose of constructing the limiting object for a general curve.

Before proceeding to describe our limiting object, it should be mentioned here that there exist general constructions for limit objects of linear series, such as the one published in an Inventiones paper by Eisenbud and Harris (cf. [6]) and an unpublished one [12]. The Eisenbud–Harris construction is extended and studied in much detail for the case of the canonical series, especially on curves comprised of two components intersecting in generically positioned points, in an Inventiones paper by Esteves and Medeiros [7] (which also references [12]). But this limit linear series is not a good limiting object from our perspective, because it does not vary smoothly, i.e. it jumps, in families (even locally trivial ones), and therefore does not appear to be useful for the sort of enumerative applications that we have in mind [13], see below.

The limiting object that we propose here is called the *sepcanonical system*. It is essentially a part of the full limit series associated to the canonical series (for a suitable modification of the family): more specifically, the largest part that never jumps. To define the sepcanonical system the curve first has to he endowed with some additional structure, closely related to Mainò's notion of enrichment [10], which we call an *azimuthal structure*, and which consists essentially of a collection of smoothing directions at separating pairs of nodes (so-called 'biseps'). The composite object is called an *azimuthal curve*, and the parameter space for these is a certain blowup of the moduli space, in which the locus of 2-separable curves becomes a divisor. This notion is best behaved for curves of 'semicompact type', meaning that distinct separating pairs of nonseparating nodes (called 'proper biseps') are disjoint, because then the smoothing directions may be chosen independently. Things are more complicated for non-semicompact-type curves, but this turns out not to be much of a problem because we shall see that in a number of specific ways, those curves behave non-hyperelliptically.

Given an azimuthal curve X_0, the sepcanonical system $|\omega_{X_0}|^{\mathrm{sep}}$ is defined as a collection of linear systems $|\omega_{X_0}|_Y^{\mathrm{sep}}$ containing $|\omega_{X_0}|_Y$, one on each '2-component' Y of X_0, i.e. subcurve obtained by