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# Height-zero characters and normal subgroups in $p$-solvable groups

A. Laradji

*Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia* [1]

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ABSTRACT

Fix a prime number $p$, and let $N$ be a normal subgroup of a finite $p$-solvable group $G$. Let $b$ be a $p$-block of $N$ and suppose $B$ is a $p$-block of $G$ covering $b$. Let $D$ be a defect group for the Fong–Reynolds correspondent of $B$ with respect to $b$ and let $\widehat{B}$ be the unique $p$-block of $NN_G(D)$ having defect group $D$ and inducing $B$. Suppose, further, that $\mu \in \mathrm{Irr}(b)$, and let $\mathrm{Irr}_0(B|\mu)$ be the set of irreducible characters in $B$ of height zero that lie over $\mu$. We show that the number of characters in $\mathrm{Irr}_0(B|\mu)$ is equal to the number of characters in $\bigcup_t \mathrm{Irr}_0(\widehat{B}|\mu^t)$, where $t$ runs through the inertial group $T$ of $b$ in $G$. This result generalizes a theorem of T. Okuyama and M. Wajima, which confirms the Alperin–McKay conjecture for $p$-solvable groups.



## 1. Introduction

Fix a prime $p$ and let $G$ be an arbitrary finite group. The McKay conjecture claims that, if $P$ is a Sylow $p$-subgroup of $G$, then $G$ and $N_G(P)$ have equal numbers of irreducible (complex) characters of degree not divisible by $p$. Although this conjecture has been verified for many families of groups, no general proof has yet been discovered. Perhaps, the most significant achievement concerning this conjecture in recent years is a work by I.M. Isaacs, G. Malle and G. Navarro [6], in which they reduced the McKay conjecture to a question about simple groups. Also, several strengthenings of the McKay conjecture due to Isaacs and Navarro [5], Navarro [14] and A. Turull [18] have been proposed with the hope of gaining some insight into the deeper underlying reason behind the conjecture.

Let now $B$ be a $p$-block of $G$. Denote by $\mathrm{Irr}_0(B)$ the set of irreducible characters in $B$ of height zero. The Alperin–McKay conjecture asserts that, if $D$ is a defect group of $B$ and $\widehat{B}$ is the $p$-block of $N_G(D)$ which corresponds to $B$ under the Brauer correspondence, then the numbers of characters in

*E-mail address:* alaradji@ksu.edu.sa.
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$\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(\widehat{B})$ are equal. Note that the Alperin–McKay conjecture implies the McKay conjecture by simply summing over all of the $p$-blocks of maximal defect. The Alperin–McKay conjecture has been shown to be valid for many families of groups. It was first proved for all $p$-solvable groups by T. Okuyama and M. Wajima in [15]. In [17], B. Späth proves a reduction for the Alperin–McKay conjecture in the same spirit as that for the McKay conjecture in [6].

Let $N$ be a normal subgroup of $G$ and let $b$ be a $p$-block of $N$ covered by $B$. Following [10], a defect group $D$ of $B$ is an *inertial defect group* of $B$ (with respect to $b$) if it is a defect group for the Fong–Reynolds correspondent of $B$ with respect to $b$.

Now let $\mu \in \mathrm{Irr}(b)$. Write $\mathrm{Irr}_0(B|\mu)$ for the intersection $\mathrm{Irr}_0(B) \cap \mathrm{Irr}(G|\mu)$, where $\mathrm{Irr}(G|\mu)$ is the set of irreducible characters of $G$ lying over $\mu$. By [10, Theorem 4.4(i)], $\mathrm{Irr}_0(B|\mu) \neq \emptyset$ if and only if $\mu$ is of height zero and $\mu$ extends to $DN$ for some inertial defect group $D$ of $B$. Also, when $N = 1$, then $\mu$ is the trivial character of $N$ and $\mathrm{Irr}_0(B|\mu)$ is the set of irreducible characters in $B$ of height zero. The aim of this paper is to prove the following generalization of Okuyama–Wajima's result [15].

**Theorem A.** *Let $N$ be a normal subgroup of a $p$-solvable group $G$, and let $B$ and $b$ be $p$-blocks of $G$ and $N$ respectively such that $B$ covers $b$. Write $T$ for the inertial group of $b$ in $G$ and let $D$ be an inertial defect group of $B$ (with respect to $b$). If $\mu \in \mathrm{Irr}(b)$ and $\widehat{B}$ is the unique $p$-block of $NN_G(D)$ with defect group $D$ such that $\widehat{B}^G = B$, then $|\mathrm{Irr}_0(B|\mu)| = |\bigcup_{t\in T} \mathrm{Irr}_0(\widehat{B}|\mu^t)|$.*

Of course, by taking $N = 1$ in Theorem A, we recover Okuyama–Wajima's theorem. We should, nevertheless, mention that our proof of Theorem A depends on this result of Okuyama and Wajima.

The equality in Theorem A involves a union of sets of characters which is certainly not in general a disjoint union. A disjoint union can be obtained as follows. Let $NN_T(D)$ act by conjugation on the set $\{\mu^t\colon\ t \in T\}$, and choose $\mu_1 = \mu, \ldots, \mu_n$, a complete set of representatives for the resulting orbits. Then it is not hard to check that $\bigcup_{i=1}^n \mathrm{Irr}_0(\widehat{B}|\mu_i)$ is a disjoint union, which equals $\bigcup_{t\in T} \mathrm{Irr}_0(\widehat{B}|\mu^t)$.

Finally, we mention that the analogue of Theorem A for Brauer characters is proved in [8].

## 2. Navarro nuclei and vertices

Let $\pi$ be a set of primes and let $\pi'$ be the complementary set of primes. Let $G$ be a $\pi$-separable group. Following Isaacs [3, Section 2], a character $\chi \in \mathrm{Irr}(G)$ is called *$\pi$-special* provided that $\chi(1)$ is a $\pi$-number and that for every $S \lhd\lhd G$ and every irreducible constituent $\theta$ of $\chi_S$, the determinantal order $o(\theta)$ of $\theta$ is a $\pi$-number. $\pi$-Special characters were first introduced and studied by D. Gajendragadkar in [1].

In Section 2 of [3], an irreducible character $\chi$ is said to be *$\pi$-factorable* if it can be written in the form $\alpha\beta$, where $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special. If $\chi \in \mathrm{Irr}(G)$ is $\pi$-factorable, we write $\chi_\pi$ and $\chi_{\pi'}$ for the $\pi$-special and $\pi'$-special factors, respectively.

A *$\pi$-factorable normal pair* in $G$ is a pair $(N,\theta)$, where $N \lhd G$ and $\theta$ is $\pi$-factorable. We order the set $\mathcal{E}(G)$ of $\pi$-factorable normal pairs by setting $(N,\theta) \leqslant (M,\eta)$ if $N \subseteq M$ and $\theta$ lies under $\eta$. The set of maximal elements of $\mathcal{E}(G)$ is denoted by $\mathcal{E}^*(G)$.

Now let $\chi \in \mathrm{Irr}(G)$. By Corollary 2.3 in [12], there is, up to $G$-conjugacy, a unique pair $(L,\zeta) \in \mathcal{E}^*(G)$ such that $\chi$ lies over $\zeta$ and that if $(N,\theta) \in \mathcal{E}(G)$ with $\theta$ under $\chi$, there exists an element $g \in G$ such that $(N,\theta^g) \leqslant (L,\zeta)$. We call such a pair a *maximal $p$-factorable normal pair under $\chi$*.

The *nucleus via normal pairs* $(W,\gamma)$ of $\chi$ is constructed by Navarro in [12] in the following manner. If $\chi$ is $\pi$-factorable, then $(W,\gamma)$ is just $(G,\chi)$. If $\chi$ is not $\pi$-factorable, select a maximal $\pi$-factorable normal pair $(L,\zeta)$ under $\chi$. Now if $I$ is the inertial group of $\zeta$ in $G$, [12, Corollary 2.4] implies that $I < G$. Let $\psi \in \mathrm{Irr}(I|\zeta)$ be the Clifford correspondent of $\chi$. We recursively define the nucleus via normal pairs of $\chi$ to be any $G$-conjugate of any nucleus via normal pairs of $\psi$. For convenience, we will simply refer to $(W,\gamma)$ as a normal nucleus of $\chi$.

Note that the set of normal nuclei of $\chi$ is a $G$-conjugacy class of pairs. Also, if $(W,\gamma)$ is a normal nucleus for $\chi$, then $\gamma^G = \chi$ and $\gamma$ is $\pi$-factorable.

We now let $\pi = \{p\}$ (a single prime), so that $G$ is $p$-solvable. Suppose that $\chi$ has normal nucleus $(W,\gamma)$. If $Q \in \mathrm{Syl}_p(W)$ and $\delta = (\gamma_p)_Q$, then $\delta \in \mathrm{Irr}(Q)$ by [1, Proposition 6.1]. The pair $(Q,\delta)$ is said