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Associated primes of local cohomology of flat extensions with regular fibers and Σ-finite D-modules

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ABSTRACT

In this manuscript, we study the following question raised by Mel Hochster: Let (R, m, K) be a local ring and S be a flat extension with regular closed fiber. Is $\mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$ finite for every ideal $I \subset S$ and $i \in \mathbb{N}$? We prove that the answer is positive when S is either a polynomial or a power series ring over R and $\dim(R/I \cap R) \leqslant 1$. In addition, we analyze when this question can be reduced to the case where S is a power series ring over R. An important tool for our proof is the use of Σ-finite D-modules, which are not necessarily finitely generated as D-modules, but whose associated primes are finite. We give examples of this class of D-modules and applications to local cohomology.

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1. Introduction

Throughout this manuscript A, R and S will always denote commutative Noetherian rings with unit. If M is an S-module and $I \subset S$ is an ideal, we denote the i-th local cohomology of M with support in I by $H^i_I(M)$. The structure of these modules has been widely studied by several authors [7,12,14–16,18,20–22]. Among the results obtained is that the set of associated primes of $H^i_I(R)$ is finite for certain regular rings. Huneke and Sharp proved this for characteristic $p > 0$ [8]. Lyubeznik showed this finiteness property for regular local rings of equal characteristic zero and finitely generated regular algebras over a field of characteristic zero [11]. Recently Bhatt, Blickle, Lyubeznik, Singh, and Zhang proved that the local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes [1]. We point out that this property does not necessarily hold for rings that are not regular [10,24,25]. Motivated by these finiteness results, Mel Hochster raised the following related questions:

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Question 1.1. Let (R, m, K) be a local ring and S be a flat extension with regular closed fiber. Is

$$\operatorname{Ass}_S H^0_{mS}\big(H^i_I(S)\big) = \mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

Question 1.2. Let (R, m, K) be a local ring and S denote either $R[x_1, \ldots, x_n]$ or $R[\![x_1, \ldots, x_n]\!]$. Is

$$\operatorname{Ass}_S H^0_{mS}\big(H^i_I(S)\big) = \mathcal{V}(mS) \cap \operatorname{Ass}_S H^i_I(S)$$

finite for every ideal $I \subset S$ and $i \in \mathbb{N}$?

It is clear that Question 1.2 is a particular case of Question 1.1. In Proposition 6.2, we show that under minor additional hypotheses these questions are equivalent. Question 1.2 has a positive answer when R is a ring of dimension 0 or 1 of any characteristic [19]. In her thesis [23], Robbins answered Question 1.2 positively for certain algebras of dimension smaller than or equal to 3 in characteristic 0. In addition, several of her results can be obtained in characteristic $p > 0$, by working in the category $C(S, R)$ (see the discussion after Remark 2.1).

A positive answer for Question 1.1 would help to prove the finiteness of the associated primes of local cohomology modules, $H^i_I(R)$, over certain regular local rings of mixed characteristic, R. For example for

$$\frac{V[\![x, y, z_1, \ldots, z_n]\!]}{(\pi - xy)V[\![x, y, z_1, \ldots, z_n]\!]} = \left(\frac{V[\![x, y]\!]}{(\pi - xy)V[\![x, y]\!]}\right)[\![z_1, \ldots, z_n]\!],$$

where $(V, \pi V, K)$ is a complete DVR of mixed characteristic. This is, to the best of our knowledge, the simplest example of a regular local ring of ramified mixed characteristic in which the finiteness of $\operatorname{Ass}_R H^i_I(R)$ is unknown.

In this manuscript, we give a partial positive answer for Question 1.1 and 1.2. Namely:

Theorem 1.3. *Let (R, m, K) be any local ring. Let S be either $R[x_1, \ldots, x_n]$ or $R[\![x_1, \ldots, x_n]\!]$. Then, $\operatorname{Ass}_S H^0_{mS} H^i_I(S)$ is finite for every ideal $I \subset S$ such that $\dim R/I \cap R \leqslant 1$ and every $i \in \mathbb{N}$. Moreover, if $mS \subset \sqrt{I}$,*

$$\operatorname{Ass}_S H^{j_1}_{J_1} \cdots H^{j_\ell}_{J_\ell} H^i_I(S)$$

is finite for all ideals $J_1, \ldots, J_\ell \subset S$ and integers $j_1, \ldots, j_\ell \in \mathbb{N}$.

Theorem 1.4. *Let $(R, m, K) \to (S, \eta, L)$ be a flat extension of local rings with regular closed fiber such that R contains a field. Let $I \subset S$ be an ideal such that $\dim R/I \cap R \leqslant 1$. Suppose that the morphism induced in the completions $\widehat{R} \to \widehat{S}$ maps a coefficient field of R into a coefficient field of S. Then,*

$$\operatorname{Ass}_S H^0_m H^i_I(S)$$

is finite for every $i \in \mathbb{N}$.

In Theorem 1.4, the hypothesis that $\widehat{\varphi}$ maps a coefficient field of $\widehat{R}$ to a coefficient field of $\widehat{S}$ is not very restrictive. For instance, it is satisfied when L is a separable extension of K (see Remark 6.3). In particular, this holds when K is a field of characteristic 0 or a perfect field of characteristic $p > 0$.

A key part of the proof of Theorem 1.3 is the use of Σ-finite D-modules, which are directed unions of finite length D-modules that satisfy certain conditions (see Definition 3.3). One of the main properties that a Σ-finite D-module satisfies is that its set of associated primes is finite. In addition,

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