Journal of Algebra 399 (2014) 782–800

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Krull dimension and monomial orders

Gregor Kemper [a], Ngo Viet Trung [b,*]

[a] Technische Universität München, Zentrum Mathematik – M11, Boltzmannstr. 3, 85748 Garching, Germany
[b] Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam

ARTICLE INFO

Article history:
Received 11 May 2013
Available online 12 November 2013
Communicated by Bernd Ulrich

MSC:
13A02
12A30
13P10

Keywords:
Krull dimension
Weight order
Monomial order
Independent sequence
Analytically independent
Associated graded ring
Jacobson ring
Subfinite algebra

ABSTRACT

We introduce the notion of independent sequences with respect to a monomial order by using the least terms of polynomials vanishing at the sequence. Our main result shows that the Krull dimension of a Noetherian ring is equal to the supremum of the length of independent sequences. The proof has led to other notions of independent sequences, which have interesting applications. For example, we can show that $\dim R/0 : J^\infty$ is the maximum number of analytically independent elements in an arbitrary ideal J of a local ring R and that $\dim B \leqslant \dim A$ if $B \subset A$ are (not necessarily finitely generated) subalgebras of a finitely generated algebra over a Noetherian Jacobson ring.

Introduction

Let R be an arbitrary Noetherian ring, where a ring is always assumed to be commutative with identity. The aim of this paper is to characterize the Krull dimension $\dim R$ by means of a monomial order on polynomial rings over R. We are inspired of a result of Lombardi in [13] (see also Coquand and Lombardi [4,5]) which says that for a positive integer s, $\dim R < s$ if and only if for every sequence of elements $a_1, \dots, a_s$ in R, there exist nonnegative integers $m_1, \dots, m_s$ and elements $c_1, \dots, c_s \in R$ such that

* Corresponding author.
E-mail addresses: kemper@ma.tum.de (G. Kemper), nvtrung@math.ac.vn (N. Viet Trung).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.10.005

$$a_1^{m_1} \cdots a_s^{m_s} + c_1 a_1^{m_1+1} + c_2 a_1^{m_1} a_2^{m_2+1} + \cdots + c_s a_1^{m_1} \cdots a_{s-1}^{m_{s-1}} a_s^{m_s+1} = 0.$$

This result has helped to develop a constructive theory for the Krull dimension [6–8].

The above relation means that $a_1, \ldots, a_s$ is a solution of the polynomial

$$x_1^{m_1} \cdots x_s^{m_s} + c_1 x_1^{m_1+1} + c_2 x_1^{m_1} x_2^{m_2+1} + \cdots + c_s x_1^{m_1} \cdots x_{s-1}^{m_{s-1}} x_s^{m_s+1}.$$

The least term of this polynomial with respect to the lexicographic order is the monomial $x_1^{m_1} \cdots x_s^{m_s}$, which has the coefficient 1. This interpretation leads us to introduce the following notion.

Let $\prec$ be a monomial order on the polynomial ring $R[x_1, x_2, \ldots]$ with infinitely many variables. For every polynomial f we write $\mathrm{in}_\prec(f)$ for the least term of f with respect to $\prec$. Let $R[X] = R[x_1, \ldots, x_s]$. We call $a_1, \ldots, a_s \in R$ a *dependent sequence* with respect to $\prec$ if there exists $f \in R[X]$ vanishing at $a_1, \ldots, a_s$ such that the coefficient of $\mathrm{in}_\prec(f)$ is invertible. Otherwise, $a_1, \ldots, a_s$ is called an *independent sequence* with respect to $\prec$.

Using this notion, we can reformulate Lombardi's result as $\dim R < s$ if and only if every sequence of elements $a_1, \ldots, a_s$ in R is dependent with respect to the lexicographic order. Out of this reformulation arises the question whether one can replace the lexicographical monomial order by other monomial orders. The proof of Lombardi does not reveal how one can relate an arbitrary monomial order to the Krull dimension of the ring. We will give a positive answer to this question by proving that $\dim R$ is the supremum of the length of independent sequences for an arbitrary monomial order. This follows from Theorem 2.7 of this paper, which in fact strengthens the above statement. As an immediate consequence, we obtain other algebraic identities between elements of R than in Lombardi's result. Although our results are not essentially computational, the independence conditions can often be treated by computer calculations. For instance, using a short program written in MAGMA [2], the first author tested millions of examples which led to the conjecture that the above question has a positive answer [12]. The proof combines techniques of Gröbner basis theory and the theory of associated graded rings of filtrations. It has led to other notions of independent sequences which are of independent interest, as we shall see below.

Our idea is to replace the monomial order $\prec$ by a weighted degree on the monomials. Given an infinite sequence $\mathbf{w}$ of positive integers $w_1, w_2, \ldots$, we may consider $R[x_1, x_2, \ldots]$ as a weighted graded ring with $\deg x_i = w_i$, $i = 1, 2, \ldots$. For every polynomial f, we write $\mathrm{in}_{\mathbf{w}}(f)$ for the weighted homogeneous part of f of least degree. We call $a_1, \ldots, a_s \in R$ a *weighted independent sequence* with respect to $\mathbf{w}$ if every coefficient of $\mathrm{in}_{\mathbf{w}}(f)$ is not invertible for all polynomials $f \in R[X]$ vanishing at $a_1, \ldots, a_s$. Otherwise, $a_1, \ldots, a_s$ is called a *weighted dependent sequence* with respect to $\mathbf{w}$. We will see that if R is a local ring and $w_i = 1$ for all i, the sequence $a_1, \ldots, a_s$ is weighted independent if and only if the elements $a_1, \ldots, a_s$ are analytically independent, a basic notion in the theory of local rings. That is the reason why we use the terminology independent sequence for the above notions.

Let $Q = (x_1 - a_1, \ldots, x_s - a_s)$ be the ideal of polynomials of $R[X]$ vanishing at $a_1, \ldots, a_s$. Let $\mathrm{in}_\prec(Q)$ and $\mathrm{in}_{\mathbf{w}}(Q)$ denote the ideals of $R[X]$ generated by the polynomials $\mathrm{in}_\prec(f)$ and $\mathrm{in}_{\mathbf{w}}(f)$, $f \in Q$. We want to find a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\prec(Q) = \mathrm{in}_{\mathbf{w}}(Q)$. It is well known in Gröbner basis theory that this can be done if $\mathrm{in}_\prec(Q)$ and $\mathrm{in}_{\mathbf{w}}(Q)$ were the largest term or the part of largest degree of f. In our setting we can solve this problem only if $\prec$ is Noetherian, that is, if every monomial has only a finite number of smaller monomials. In this case, $a_1, \ldots, a_s$ is independent with respect to $\mathbf{w}$ if and only if it is independent with respect to $\prec$. If $\prec$ is not Noetherian, we can still find a Noetherian monomial order $\prec'$ such that if $a_1, \ldots, a_s$ is independent with respect to $\prec$, then $a_1 a_i, \ldots, a_s a_i$ is independent with respect to $\prec'$ for some index i. By this way, we can reduce our investigation on the length of independent sequences to the weighted graded case.

We shall see that for every weight sequence $\mathbf{w}$, $\mathrm{in}_{\mathbf{w}}(Q)$ is the defining ideal of the associated graded ring of certain filtration of R. Using properties of this associated graded ring we can show that the length of a weighted independent sequence is bounded above by $\dim R$, and that $a_1, \ldots, a_s$ is a weighted independent sequence if $\mathrm{ht}(a_1, \ldots, a_s) = s$. From this it follows that $\dim R$ is the supremum of the length of independent sequences with respect to $\mathbf{w}$. This is formulated in more detail in Theorem 1.8 of this paper. Furthermore, we can also show that $\dim R/\bigcup_{n \geqslant 1}(0 : J^n)$ is the supremum