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A remark on Rickard complexes

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ABSTRACT

In this paper, we characterize a Rickard complex, which induces a Rickard equivalence between the block algebras of a block b and its Brauer correspondent and whose vertices have the same order as defect groups of the block b. The homology of such a Rickard complex vanishes at all degree but degree q, and the homology at degree q induces a basic Morita equivalence between the block algebras in the sense of Puig.

1. In [5], J. Rickard exhibits a splendid Rickard complex, which induces a splendid Rickard equivalence between the block algebra of a p-block of a finite p-nilpotent group and the group algebra of its defect group, which is isomorphic to a Morita equivalence not induced by a p-permutation module. Then in [1], Harris and Linckelmann extend Rickard's technique and show a splendid Rickard complex, which induces a splendid Rickard equivalence between the block algebras of a block for finite p-solvable groups with abelian defect groups and its Brauer correspondent, which is also isomorphic to a Morita equivalence not induced by a p-permutation module. The two splendid Rickard complexes have vertex $\Delta(Q)$ in terms of Puig (see Paragraph 7 below), where Q is a defect group of the blocks and $\Delta(Q)$ is the diagonal subgroup of $Q \times Q$. In this paper, we characterize a Rickard complex, which induces a Rickard equivalence between the block algebras of a block b and its Brauer correspondent and whose vertices have the same order as defect groups of the block b.

2. We recollect some notation in [4]. Let p be a prime number and let k be an algebraically closed residue field k of characteristic p. From the point of view of [4], complexes are considered as $\mathfrak{D}$-modules where, denoting by $\mathfrak{F}$ the commutative k-algebra of all the k-valued functions on the set $\mathbf{Z}$ of all rational integers, $\mathfrak{D}$ is the k-algebra containing $\mathfrak{F}$ as a unitary k-subalgebra and an element d such that

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$$\mathfrak{D} = \mathfrak{F} \oplus \mathfrak{F}d, \qquad d^2 = 0 \quad \text{and} \quad df = \mathrm{sh}(f)d \neq 0 \quad \text{for any } f \in \mathfrak{F} - \{0\}$$

where sh denotes the automorphism on the k-algebra $\mathfrak{F}$ mapping $f \in \mathfrak{F}$ onto the k-valued function sending $z \in \mathbf{Z}$ to $f(z+1)$; moreover, we denote by s and i_z, for any $z \in \mathbf{Z}$, the k-valued functions mapping $z' \in \mathbf{Z}$ on $(-1)^{z'}$ and $\delta_z^{z'}$ respectively. Except for all the group algebras over $\mathfrak{D}$, we assume that all the modules and the algebras over k are finitely generated. If A is a k-algebra we denote by A^* the group of invertible elements of A, and by A° the opposite k-algebra. Note that we have an isomorphism $\mathrm{t} : \mathfrak{D} \cong \mathfrak{D}^\circ$ mapping $f \in \mathfrak{F}$ on the k-valued function sending $z \in \mathbf{Z}$ to $f(-z)$, and d on sd.

3. A $\mathfrak{D}$-interior algebra is a k-algebra A endowed with a unitary k-algebra homomorphism $\varrho : \mathfrak{D} \to A$; for any $x, y \in \mathfrak{D}$ and any $a \in A$, we write $x \cdot a \cdot y$ instead of $\varrho(x)a\varrho(y)$. Note that the isomorphism $\mathrm{t} : \mathfrak{D} \cong \mathfrak{D}^\circ$ then determines a $\mathfrak{D}$-interior algebra structure for A°. Moreover, we have a k-algebra homomorphism $\mathfrak{D} \to k$ mapping $f + f'd$ on $f(0)$ for any $f, f' \in \mathfrak{F}$, so that any k-algebra admits a *trivial* structure of $\mathfrak{D}$-interior algebra. The $\mathfrak{D}$-interior algebra structure on A induces a $\mathfrak{D}$-module structure on A by the equalities

$$f(a) = \sum_{z,z' \in \mathbf{Z}} f(z) i_{z'} \cdot a \cdot i_{z'-z} \quad \text{and} \quad d(a) = (d \cdot a - a \cdot d) \cdot s$$

for any $a \in A$ and any $f \in \mathfrak{F}$. The k-algebra A endowed with this $\mathfrak{D}$-module is a $\mathfrak{D}$-algebra in the sense of Puig (see [4, 11.2.4]).

4. Let G be a finite group; recall that a kG-interior algebra is a k-algebra endowed with a unitary k-algebra homomorphism from kG. Similarly, a $\mathfrak{D}G$-interior algebra is a k-algebra A endowed with a unitary k-algebra homomorphism $\rho : \mathfrak{D}G \to A$ (but A is always finitely generated!); for any $x \in \mathfrak{D}G$ and $a \in A$, we write $x{\cdot}a$ and $a{\cdot}x$ instead of $\rho(x)a$ and $a\rho(x)$ respectively. If A and A' are $\mathfrak{D}G$-interior algebras, the tensor product $A \otimes_k A'$ admits a $\mathfrak{D}G$-interior algebra structure given by

$$f{\cdot}\big(a \otimes a'\big) = \sum_{z,z' \in \mathbf{Z}} f\big(z+z'\big)\, i_z{\cdot}a \otimes i_{z'}{\cdot}a', \qquad d{\cdot}\big(a \otimes a'\big) = d{\cdot}a \otimes s{\cdot}a' + a \otimes d{\cdot}a'$$

and $g{\cdot}(a \otimes a') = g{\cdot}a \otimes g{\cdot}a'$ for any $f \in \mathfrak{F}$, any $g \in G$ and any $a, a' \in A$. Here the first equality makes sense since in the sum above all but a finite number of terms vanish and since we have $\mathrm{sh}(s) = -s$. For any subgroup H of G, we denote by A^H the centralizer of $\rho(H)$ in A; obviously $\rho(x) \in A^H$ for any $x \in \mathfrak{D}C_G(H)$ and thus the restriction of ρ to $\mathfrak{D}C_G(H)$ induces a $\mathfrak{D}C_G(H)$-interior algebra structure on A^H. Let B and C be two kG-interior algebras. A k-algebra homomorphism $f : B \to C$ is a kG-interior algebra homomorphism if f preserves the kG-interior algebra structures on B and C; furthermore, if f is injective and $f(B) = f(1)Cf(1)$, then f is a kG-interior algebra embedding. Similarly $\mathfrak{D}G$-interior algebra homomorphisms and $\mathfrak{D}G$-interior algebra embeddings are defined.

5. Let us denote by $\mathbb{C}_0(A)$ the centralizer of the image of $\mathfrak{D}$ in A; since the images of $\mathfrak{D}$ and G centralize each other, $\mathbb{C}_0(A)$ inherits a kG-interior algebra structure and, according to the terminology in [4], the pointed groups, their inclusions, the local pointed groups, etc. over the $\mathfrak{D}G$-interior algebra A are nothing but the pointed groups, their inclusions, the local pointed groups, etc. over the kG-interior algebra $\mathbb{C}_0(A)$. However, if H_β is a pointed group over A, so that β is a conjugacy class of primitive idempotents in $\mathbb{C}_0(A)^H$, the k-algebra $A_\beta = iAi$ for any $i \in \beta$ inherits a $\mathfrak{D}H$-interior algebra structure mapping $y \in \mathfrak{D}H$ on $y{\cdot}i = i{\cdot}y$; and the k-algebra A_β is called an embedded algebra associated with H_β. For any subgroup H of G, we call contractible any point contained in the two-sided ideal

$$\mathbb{B}_0\big(A^H\big) = \mathbb{C}_0(A)^H \cap \big\{d{\cdot}a + a{\cdot}d \,\big|\, a \in A^H\big\}$$