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Strongly clean matrices over arbitrary rings

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ABSTRACT

We characterize when the companion matrix of a monic polynomial over an arbitrary ring R is strongly clean, in terms of a type of ideal-theoretic factorization (which we call an iSRC factorization) in the polynomial ring $R[t]$. This provides a nontrivial necessary condition for $\mathbb{M}_n(R)$ to be strongly clean, for R arbitrary. If the ring in question is either local or commutative, then we can say more (generalizing and extending most of what is currently known about this problem). If R is local, our iSRC factorization is equivalent to an actual polynomial factorization, generalizing results in [1], [18] and [12]. If, instead, R is commutative and $h \in R[t]$ is monic, we again show that an iSRC factorization yields a polynomial factorization, and we prove that h has such a factorization if and only if its companion matrix is strongly clean, if and only if every algebraic element (in every R-algebra) which satisfies h is strongly clean. This generalizes the work done in [1] on commutative local rings and provides a characterization of strong cleanness in $\mathbb{M}_n(R)$ for any commutative ring R.

1. Introduction

Recall that, following [13], an element x of a ring R is said to be *strongly clean* if there exists an idempotent $e \in R$ and a unit $u \in R$ for which $x = e + u$ and $eu = ue$; a ring is said to be strongly clean if each of its elements is strongly clean. The class of strongly clean rings contains many familiar examples. As noted in [13], strongly clean rings generalize the classical strongly π-regular rings. In particular, any strongly clean endomorphism of a module satisfies a generalized version of Fitting's Lemma (which we recall in Lemma 2.1). Additionally, every local ring is strongly clean. In the

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commutative case, the strongly clean rings are precisely the exchange rings (in general, the exchange condition is weaker than strong cleanness). There has been considerable interest in the construction of examples of strongly clean rings, and many articles have been written on the subject over the last decade.

In the present article, we focus on strong cleanness of matrix rings. We begin with some motivation. In [13], Nicholson poses several open questions about strongly clean rings. Notably for this article, he asks whether strong cleanness is a Morita invariant. Although strong cleanness does pass to (Peirce) corner rings (proved independently by T.Y. Lam and E. Sánchez-Campos, and also appearing in [2] and [4]), it was shown in both [16] and [14] (in both cases, using the example $\mathbb{M}_2(\mathbb{Z}_{(2)})$) that a matrix ring over a commutative local ring need not be strongly clean. This discovery led many authors to consider the problem of determining when a matrix ring over a strongly clean ring is, itself, strongly clean (since strong cleanness passes to corners, it is necessary that the base ring be strongly clean). This paper continues this line of inquiry.

We begin by giving a brief history of the progress that has been made on this problem. To provide some structure to our presentation, we organize our account according to the base rings for which the problem has been investigated as well as according to the techniques employed. We begin with the first articles which tackled the problem, which constitute what we will call the "element-theoretic" approach. In this early work, the base rings under consideration were both commutative and local. This is sensible since local rings are the simplest strongly clean rings (for example, they are the strongly clean rings with no nontrivial idempotents), and the restriction to commutative rings provides obvious advantages when working with matrices (e.g. the Cayley–Hamilton Theorem). The first two characterization results were proved in [5], in which the authors characterized the commutative local rings R for which $\mathbb{M}_2(R)$ is strongly clean, and in [11], in which the methods of [5] were completed to obtain an analogous elementwise characterization for $\mathbb{M}_2(R)$, also in the case where R is commutative local. Additional auxiliary results of this nature are proved in [4], [6] and [17]. The characterizations in [5] and [11] are stated in terms of roots of degree two polynomials, and the methods employed rely on an explicit equational characterization of the units and idempotents of the matrix ring $\mathbb{M}_2(R)$. This motivates our designation of this approach as "element-theoretic". However, these techniques are inherently limited, and we will see that the results obtained with this approach are specific to the case of 2×2 matrices.

At about the same time, independently in [1], the present authors, together with G. Borooah, approached the problem of strongly clean matrices using Nicholson's module-theoretic characterization of strong cleanness (reproduced below in Lemma 2.1). We will therefore refer to this approach as "module-theoretic". In [1], the authors view a matrix in $\mathbb{M}_n(R)$ as an endomorphism of the free module R^n and exploit the fact that local rings are projective-free in order to view strong cleanness in terms of direct sum decompositions of R^n. Examining the behavior of the characteristic polynomial of such an endomorphism, with respect to this decomposition, the authors of [1] both generalize and provide context for the previous results. The main result of [1] characterizes strong cleanness of $\mathbb{M}_n(R)$ (R commutative local, n arbitrary) in terms of a type of factorization, called an *SRC* factorization, of monic polynomials in $R[t]$. The analysis in [1] illustrates that the situation in the 2×2 case is rather special (see, for example [1, Proposition 30 and Theorem 37]) and that, in general, strong cleanness of a single matrix is not determined by factorizations of its characteristic polynomial (see [1, Example 10]). What can be done, elementwise, is to characterize when a companion matrix is strongly clean in terms of a factorization of its characteristic polynomial, and this is precisely what an SRC factorization does. Finally, [1] provides examples of strongly clean matrix rings, showing in particular (see [1, Corollary 21]) that $\mathbb{M}_n(R)$ is strongly clean for any Henselian local ring R (a fact noted independently and by different methods, by F. Couchot in [3, Theorem 1]).

We remark briefly that [3] contains several useful results concerning strong cleanness of matrix rings over commutative local rings (as well as other rings, for example direct limits of module-finite algebras over commutative local rings). For the most part, [3] investigates the relationship between Henselianness (or local Henselianness) and strong cleanness of algebras. In particular, Couchot conjectures that, for R commutative local, $\mathbb{M}_n(R)$ is strongly clean for all n if and only if R is Henselian (this was also asked independently in [1, Problem 29]), and he proves that this is true in several cases