Journal of Algebra 399 (2014) 870–886

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On infinitely generated groups whose proper subgroups are solvable

A.O. Asar

ARTICLE INFO

Article history:
Received 7 May 2013
Available online 18 November 2013
Communicated by E.I. Khukhro

Dedicated to the memory of Brian Hartley

Keywords:
Minimal non-solvable
Minimal non-hypercentral
Minimal non-FC
Residually nilpotent
Isolator

ABSTRACT

In this work infinitely generated groups are considered whose proper subgroups are solvable and in whose homomorphic images finitely generated subgroups have residually nilpotent normal closures. It is shown that a periodic group with this property is locally finite and either solvable or is a locally nilpotent p-group and has a homomorphic image which is a perfect Fitting group with additional properties. However if "residually nilpotent" is replaced by "residually (finite and nilpotent)", then the group is solvable. Furthermore if G is non-periodic and locally nilpotent, then the group is solvable without the hypothesis on normal closures.

1. Introduction

(In this work p always denotes a prime number.) The following two results were obtained in [5]. Let G be a locally finite p-group. (a) If G is a Fitting group whose proper subgroups are hypercentral and solvable, in particular if G is barely transitive and a point stabilizer is hypercentral and solvable, then G cannot be perfect; (b) if all the proper subgroups of G are nilpotent-by-Chernikov, then G is nilpotent-by-Chernikov. Since then these results have been used and generalized in some papers (see, for example, [1, Theorem 1.1], [2, Theorem 1.2], [3], [6, Theorem 1.1], [8, Theorem 2] and [9, Theorem 1.1]). The results given in (a) and (b) above are special cases of the following general problem.

Problem. Let G be an infinitely generated group whose proper subgroups are solvable. Is G then solvable?

The answer to this general problem is not known and might be difficult. However it is well known that there are infinite simple groups whose proper subgroups are solvable. Ol'shanskii in [19] constructed infinite simple p-groups for sufficiently large primes p in which every proper subgroup is

E-mail address: aliasar@gazi.edu.tr.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.001

cyclic of order p. But these groups are finitely generated. The aim of this work is to consider the following rather special case of the above problem. *Let G be an infinitely generated group whose proper subgroups are solvable and in whose homomorphic images finitely generated subgroups have residually nilpotent normal closures. Is G then solvable?* Here it is shown that in the periodic case G is locally finite and either solvable or is a perfect locally nilpotent p-group and has a homomorphic image which is a Fitting group with additional properties. (Theorem 1.4). However if "residually nilpotent" is replaced by "residually (finite and nilpotent)" in Theorem 1.4, then the group is solvable (Corollary 1.5). Furthermore if G is non-periodic and locally nilpotent, then G is solvable without the hypothesis on normal closures, as was observed by the referee (Corollary 1.7). Theorem 1.1(a) and Corollary 1.6 generalize the results given in (a) above and [1, Theorem 1.1], respectively. Finally we note that Theorem 1.4 and Corollary 1.6 are partial answers to Problem 16.5(b) and (c) in the Kourovka Notebook (Unsolved Problems in Group Theory by Khukhro and Mazurov).

Let G be a group and let P be a group theoretical property. If every proper subgroup of G satisfies P but G itself does not satisfy P, then G is called a **minimal non-**P**-group** (*MNP*-**group** for short). For example P may stand for "solvable", "hypercentral", "finite conjugacy class" and then the group may be called as *MNS*-**group**, *MNHC*-**group** or *MNFC*-**group**, respectively.

In this work the main approach to the problem stated above is to try to exploit the property that was very useful in [4,5]. For this reason it will be suitable to begin by stating it here. Let G be a group and let Y be a subset of G.

(∗) For every element $w \neq 1$ and for every finitely generated subgroup U of G with $w \notin U$ there exists a finitely generated subgroup V of G containing U and a proper subgroup L of G such that

$$w \notin V \text{ but } w \in \langle V, y\rangle \quad \text{for every } y \in Y \setminus L$$

In this case G is said to satisfy the property (∗) with respect to Y. It follows from Proposition A below that (∗) is satisfied in a perfect locally finite minimal non-hypercentral group (see also [4, Lemma 2.2] or [5, Lemma 2.3]). More generally it follows from [5, Lemma 2.3] that if Y is a generating subset of G such that every proper subgroup of G generated by subsets of Y is hypercentral, then G satisfies (∗) with respect to Y. Note that a perfect locally finite minimal non-FC-group is a p-group by [7, Theorem 2] and [14, Theorem] (see also [22, Lemma 8.15]), in which case every proper subgroup is hypercentral and so the group satisfies (∗). Also a perfect barely transitive group satisfies (∗) by Lemma 2.8 below.

We now extend the (∗) definition as follows. Let $1 \neq w \in G$ and V be a finitely generated subgroup of G with $w \notin V$. Then the pair (w, V) is called a (∗)-**pair** with respect to Y or for Y, if it satisfies (∗) with respect to Y; that is, if there exists a proper subgroup L of G such that $w \notin V$ but $w \in \langle V, y\rangle$ for every $y \in Y \setminus L$. In this case the triple (w, V, L) is also called a (∗)-**triple**. Of course if in the above definitions Y is not specified, then $Y = G$. Note that (w, V, L) is a (∗)-triple is equivalent to the property that $\bigcap_{y \in Y \setminus L} \langle V, y\rangle \neq V$. In [1–3] non-trivial conditions are obtained for the existence of (∗)-triples.

Again let w be an element and V be a finitely generated subgroup of a group G with $w \notin V$. A subgroup E of G which is maximal with respect to the condition that $w \notin E$ but $V \leqslant E$ is called a (w, V)-**maximal subgroup** of G. Furthermore the pair (w, V) is called a **distinguished pair** for G, if there exists no (∗)-pair (w, U) with respect to G with $V \leqslant U$ and if

$$d\big(\langle V, y\rangle\big) > d(V) \quad \text{implies that} \quad w \in \langle V, y\rangle \quad \text{for every } y \in G$$

where $d(V)$ denotes the derived length of V. Put

$$E(w, V) = \big\{y \in G\colon\ w \notin \langle V, y\rangle\big\}$$

$$E^*(w, V) = \big\{E\colon\ E \text{ is an } (w, V)\text{-maximal subgroup of } G\big\}$$