Journal of Algebra 399 (2014) 894–903

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Uniserial dimension of modules

Z. Nazemian [a], A. Ghorbani [a,*], M. Behboodi [a,b,1]

[a] Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran
[b] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

ARTICLE INFO

Article history:
Received 13 July 2013
Available online 18 November 2013
Communicated by Louis Rowen

MSC:
primary 16D70, 16D90, 16P70
secondary 03E10, 13E05

Keywords:
Uniserial module
Uniserial dimension
Semisimple module
Semisimple Artinian ring

ABSTRACT

Until now there has been no suitable dimension to measure how far a module deviates from being uniserial. We define and study a new dimension, which we call uniserial dimension. The uniserial dimension is a measure of how far a module deviates from being uniserial. It is shown that for a ring R and an ordinal number α, there exists an R-module of uniserial dimension α. We show that a commutative ring R is Noetherian (resp. Artinian) if and only if every finitely generated R-module has (resp. finite) uniserial dimension. We characterize rings whose modules have uniserial dimension. In fact, it is shown that every right R-module has uniserial dimension if and only if the free right R-module $\bigoplus_{i=1}^{\infty} R$ has uniserial dimension if and only if R is a semisimple Artinian ring.

0. Introduction

By a *valuation ring* (also called a *chain ring*) we mean a commutative ring R whose ideals are totally ordered by inclusion. A valuation ring that is a domain is called *valuation domain*. The notion of valuation domain has been generalized in various directions. One is the Manis valuation that furnishes some commutative non-domains with a kind of valuation. The other is concerned with (non-commutative) chain rings. Note that by definition a *right chain ring* R has a linearly ordered lattice of right ideals. Left chain rings are defined similarly. Uniserial modules are immediate generalization of right chain rings. A module N is called *uniserial* if its submodules are linearly ordered by inclusion. Clearly, every submodule and every factor module of a uniserial module is uniserial. Artinian serial rings were probably studied first in 1935 by Köthe and the term uniserial is due to him. The structure

* Corresponding author.
E-mail addresses: z.nazemian@math.iut.ac.ir (Z. Nazemian), a_ghorbani@cc.iut.ac.ir (A. Ghorbani), mbehbood@cc.iut.ac.ir (M. Behboodi).

[1] The third author was financially supported by the IPM, Iran.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.054

of these rings was then studied by Asano in [2], Nakayama in [10], who called generalized uniserial ring what we call Artinian serial ring, Fuller in [6], Eisenbud and Griffith in [4], and by Warfield in [14], who defined serial rings in such a way that they need not be Noetherian. Hence the definition of (right) serial rings we use in this article follows Warfield's terminology. A *serial* module is a module that is a direct sum of uniserial modules, and a ring R is *serial* if the two modules R_R and ${}_RR$ are both serial modules. Important classes of rings yield examples of serial rings. For instance, semisimple Artinian rings and rings of triangular matrices over a field are serial rings.

Although uniserial modules are uniform and there are some dimensions to show how far the module is from being uniform, there is no dimension to show how far the module is from being uniserial. For example although the $\mathbb{Z}$-module $\mathbb{Z}$ is uniform, it is too far from being uniserial.

Throughout this paper, let R denote an arbitrary ring with identity. All modules are assumed to be unitary. If N is a submodule (resp. proper submodule) of M we write $N \leqslant M$ (resp. $N < M$). For a module M_R we write $\mathrm{soc}(M)$ and $\mathrm{Rad}(M)$, for the socle and the Jacobson radical of M, respectively. Also $\mathrm{J}(R)$ will be used for the Jacobson radical of a ring R.

In this article we define and study uniserial dimension of modules. It is an ordinal-valued invariant that measures how far a module is from being uniserial. It is convenient to begin our list of ordinals with 1. In order to define uniserial dimension for modules over a ring R, we first define, by transfinite induction, classes ζ_α of R-modules for all ordinals $\alpha \geqslant 1$. To start with, let ζ_1 be the class of non-zero uniserial modules. Next, consider an ordinal $\alpha > 1$; if ζ_β has been defined for all ordinals $\beta < \alpha$, let ζ_α be the class of those R-modules M such that, for every submodule $N < M$, where $M/N \ncong M$, we have $M/N \in \bigcup_{\beta<\alpha} \zeta_\beta$. If an R-module M belongs to some ζ_α, then the least such α is the *uniserial dimension of* M, denoted $\mathrm{u.s.dim}(M)$. For $M = 0$, we define $\mathrm{u.s.dim}(M) = 0$. If $M \neq 0$ and M does not belong to any ζ_α, then we say that M has no uniserial dimension.

In Section 1, we consider some basic properties of the uniserial dimension. In particular, we show that an R-module M has uniserial dimension if and only if for every ascending chain $M_1 \leqslant M_2 \leqslant \cdots$ of submodules of M, there exists $n \geqslant 1$ such that M/M_n is uniserial or $M/M_n \cong M/M_k$ for all $k \geqslant n$ (see Proposition 1.3). This yields that every Noetherian module has uniserial dimension. We also show that a commutative ring R is Noetherian (resp. Artinian) if and only if every finitely generated R-module has (resp. finite) uniserial dimension (see Corollaries 1.6 and 1.16). Recall that a semisimple module M is said to be *homogeneous* if M is a direct sum of pairwise isomorphic simple submodules. We see in Proposition 1.18 that a semisimple module has uniserial dimension if and only if it is a finite direct sum of homogeneous semisimple modules. Moreover the uniserial dimension of a finitely generated semisimple module is equal to its length.

In Section 2, it is shown that for a ring R the right module $\bigoplus_{i=1}^{\infty} R$ has uniserial dimension if and only if R is semisimple Artinian if and only if; all right R-modules have uniserial dimension (see Theorem 2.6).

1. Uniserial dimension

Let M be a non-zero R-module. Hirano and Mogami proved in [8, Proposition 1] that if, for every proper submodule N of M, $M/N \cong M$ then M is uniserial and Artinian. Moreover, the lattice of submodules of M, which must therefore be order-isomorphic to an ordinal α, must be an ordinal α of a very special type. Actually M is simple or $\alpha = \omega^\beta + 1$ for some ordinal β, where ω denotes the first limit ordinal number. Hirano and Mogami also proved in [8, Theorem 8] that if R is commutative, then the following conditions are equivalent for an R-module M: (a) $M/N \cong M$ for every proper submodule N of M; (b) the lattice of submodules of M is order-isomorphic to 1 or $\omega + 1$. They even didn't know whether a module M with $M/N \cong M$ for every proper submodule N of M and the lattice of submodules of M order-isomorphic to $\omega^2 + 1$ could exist (see [8, Question 7]), a fact that was later proved by Facchini and Salce in [5, Section 2]. All these facts motivate us to introduce the notion of uniserial dimension for modules:

Definition 1.1. In order to define uniserial dimension for modules over a ring R, we first define, by transfinite induction, classes ζ_α of R-modules for all ordinals $\alpha \geqslant 1$. To start with, let ζ_1 be the class of non-zero uniserial modules. Next, consider an ordinal $\alpha > 1$; if ζ_β has been defined for all ordinals