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Curtis–Tits groups generalizing Kac–Moody groups of type $\widetilde{A}_{n-1}$

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ABSTRACT

In [13] we define a Curtis–Tits group as a certain generalization of a Kac–Moody group. We distinguish between orientable and non-orientable Curtis–Tits groups and identify all orientable Curtis–Tits groups as Kac–Moody groups associated to twin-buildings.
In the present paper we construct all orientable as well as non-orientable Curtis–Tits groups with diagram $\widetilde{A}_{n-1}$ ($n \geqslant 4$) over a field $\Bbbk$ of size at least 4. The resulting groups are quite interesting in their own right. The orientable ones are related to Drinfeld's construction of vector bundles over a non-commutative projective line and to the classical groups over cyclic algebras. The non-orientable ones are related to expander graphs [14] and have symplectic, orthogonal and unitary groups as quotients.

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1. Introduction

The theory of the infinite dimensional Lie algebras called Kac–Moody algebras was initially developed by Victor Kac and Robert Moody. The development of a theory of Kac–Moody groups as analogues of Chevalley groups was made possible by the work of Kac and Peterson. In [44] J. Tits gives an alternative definition of a group of Kac–Moody type as being a group with a twin-root datum, which implies that they are symmetry groups of Moufang twin-buildings.

In [2] P. Abramenko and B. Mühlherr generalize a celebrated theorem of Curtis and Tits on groups with finite BN-pair [18,42] to groups of Kac–Moody type. This theorem states that a Kac–Moody group

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$\mathbf{G}$ is the universal completion of an amalgam of rank two (Levi) subgroups, as they are arranged inside $\mathbf{G}$ itself. This result was later refined by Caprace [16]. Similar results on Curtis–Tits–Phan type amalgams have been obtained in [7,6,8,11,12,23,27,29,24]. For an overview of that subject see Köhl [26].

In order to describe the main result from [13] we introduce some notation. Let k be a (commutative) field of order at least 4. Let Γ be a connected simply-laced Dynkin diagram over an index set I without triangles. For any $J \subseteq I$, let Γ_J be the subdiagram supported by the node set J. In [13] we take the Curtis–Tits type results as a starting point and define a *Curtis–Tits amalgam* with diagram Γ over k to be an amalgam of groups such that the sub-amalgam corresponding to a two-element subset $J \subseteq I$ is the amalgam of derived groups of standard Levi subgroups of some rank-2 group of Lie type Γ_J over k. There is no a priori reference to an ambient group, nor to the existence of an associated (twin-)building. Indeed, there is no a priori guarantee that the amalgam will not collapse. Also, this definition clearly generalizes to other Dynkin diagrams.

We then classify all Curtis–Tits amalgams with diagram Γ over k using the following data (for similar results in special cases see [22,25]). Viewing Γ as a graph, for $i_0 \in I$, let $\pi(\Gamma, i_0)$ denote the (first) fundamental group of Γ with base point i_0. Also we let the group $\mathrm{Aut}(\mathsf{k}) \times \langle\tau\rangle$ (with τ of order 2) act as a subgroup of the stabilizer in $\mathrm{Aut}(\mathrm{SL}_2(\mathsf{k}))$ of a fixed torus in $\mathrm{SL}_2(\mathsf{k})$; τ denotes the transpose-inverse map with respect to that torus. The main result of [13] is the following.

Classification Theorem. *There is a natural bijection between isomorphism classes of Curtis–Tits amalgams with diagram Γ over the field k and group homomorphisms $\Theta : \pi(\Gamma, i_0) \to \langle\tau\rangle \times \mathrm{Aut}(\mathsf{k})$.*

We call amalgams corresponding to homomorphisms Θ whose image lies inside $\mathrm{Aut}(\mathsf{k})$ "orientable"; others are called "non-orientable". It is not at all immediate that all non-orientable amalgams arising from the Classification Theorem are non-collapsing, i.e. that their universal completion is non-trivial. We shall call a non-trivial group a *Curtis–Tits group* if it is the universal completion of a Curtis–Tits amalgam. It is shown that orientable Curtis–Tits amalgams are precisely those arising from the Curtis–Tits theorem applied to a group of Kac–Moody type. Thus, groups of Kac–Moody type are orientable Curtis–Tits groups.

1.1. Main results

We now specify Γ to be the Dynkin diagram of type $\widetilde{A}_{n-1}$ labeled cyclically with index set $I = \{1, 2, \ldots, n\}$, where $n \geqslant 4$. The purpose of the present paper is to construct all orientable and non-orientable Curtis–Tits groups over k with diagram Γ and to study their properties.

The paper is structured as follows. In Section 2 we introduce the relevant notions about amalgams and describe all possible Curtis–Tits amalgams of type Γ over k. For each $\delta \in \mathrm{Aut}(\mathsf{k}) \times \langle\tau\rangle$ we introduce a Curtis–Tits amalgam $\mathscr{G}^\delta$ corresponding to δ via Θ as in the Classification Theorem and denote its universal completion $(\tilde{\mathbf{G}}^\delta, \tilde{\phi}^\delta)$. In Section 3 we exhibit a non-trivial completion for orientable Curtis–Tits groups using a description of the corresponding twin-building. In order to state the main result of this section we introduce the following notation. For $\alpha \in \mathrm{Aut}(\mathsf{k})$, let $\mathsf{R}_\alpha = \mathsf{k}\{t, t^{-1}\}$ be the ring of skew Laurent polynomials with coefficients in the field k such that for $x \in \mathsf{k}$ we have $txt^{-1} = x^\alpha$. Let k_α be the fixed field of α in k. We use the Dieudonné determinant to identify $\mathrm{SL}_n(\mathsf{R}_\alpha)$. As usual, the center of a group X, is denoted $Z(X)$. We obtain the following.

Theorem 1. *For $\alpha \in \mathrm{Aut}(\mathsf{k})$, the universal completion $\tilde{\mathbf{G}}^\alpha$ of $\mathscr{G}^\alpha$ is an extension of $\mathrm{SL}_n(\mathsf{R})$ by a subgroup H of the center $Z(\tilde{\mathbf{G}}^\alpha)$, which is isomorphic to a subgroup of k_α^*.*

In Section 4 we consider the case $\delta = \alpha\tau$ for some $\alpha \in \mathrm{Aut}(\mathsf{k})$ and exhibit a non-trivial completion of $\mathscr{G}^\delta$. Via Proposition 4.7 we obtain the first two parts of Theorem 2 below. Demonstrating the universality and identification of the completion is more involved this time and takes up Subsections 4.3, 4.4, 4.5 and 4.7.

In order to state the main result of Section 4, we introduce the following notation. Let σ be the automorphism of R_{α^2} inducing α^{-1} on k and interchanging t and t^{-1} and let β be the asymmet-