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Invariants of centralisers in positive characteristic

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ABSTRACT

Let Q be a simple algebraic group of type A or C over a field of good positive characteristic. Let $x \in \mathfrak{q} = \mathrm{Lie}(Q)$ and consider the centraliser $\mathfrak{q}_x = \{y \in \mathfrak{q}\colon [xy] = 0\}$. We show that the invariant algebra $S(\mathfrak{q}_x)^{\mathfrak{q}_x}$ is generated by the pth power subalgebra and the mod p reduction of the characteristic zero invariant algebra. The latter algebra is known to be polynomial [17] and we show that it remains so after reduction. Using a theory of symmetrisation in positive characteristic we prove the analogue of this result in the enveloping algebra, where the p-centre plays the role of the pth power subalgebra. In Zassenhaus' foundational work [30], the invariant theory and representation theory of modular Lie algebras were shown to be explicitly intertwined. We exploit his theory to give a precise upper bound for the dimensions of simple $\mathfrak{q}_x$-modules. An application to the geometry of the Zassenhaus variety is given.
When $\mathfrak{g}$ is of type A and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a symmetric decomposition of orthogonal type we use similar methods to show that for every nilpotent $e \in \mathfrak{k}$ the invariant algebra $S(\mathfrak{p}_e)^{\mathfrak{k}_e}$ is generated by the pth power subalgebra and $S(\mathfrak{p}_e)^{K_e}$ which is also shown to be polynomial.

1. Introduction

In [17] Premet asked the following question: *if* Q *is a reductive group over* $\mathbb{C}$ *and* $x \in \mathfrak{q}$ *then is* $S(\mathfrak{q}_x)^{Q_x}$ *a polynomial algebra on* $\mathrm{rank}(\mathfrak{q})$ *variables*? The authors of [17] answered this question in the affirmative for simple groups of type A and C. Some partial results were obtained in types B and D. In a completely separate work [1] Brown and Brundan proved the statement again in type A using very different methods. A counterexample was found by Yakimova in [28]; here x is a long root vector in type E_8. Very recently a counterexample has been given by Charbonnel and Moreau in type D_7 [4]. Making use of the theory of finite W-algebras and the techniques of reduction modulo p the authors of the latter article were able to provide a very general criterion on $x \in \mathfrak{q}$ for $S(\mathfrak{q}_x)^{\mathfrak{q}_x}$ to be a polynomial algebra.

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The purpose of this article is to bring the above invariant theoretic discussion into the characteristic p realm, and exploit some combinatorial techniques to study the representation theory of the centralisers $\mathfrak{q}_x$ in type A and C. When an algebraic group may be reduced modulo p, in an appropriate sense, we have groups Q, Q_p and their respective Lie algebras $\mathfrak{q}$, $\mathfrak{q}_p$. If Q is reductive and the characteristic of the field is very good for Q then it is known that $S(\mathfrak{q}_p)^{\mathfrak{q}_p}$ is generated by $S(\mathfrak{q}_p)^p$ and a natural choice of mod p reduction of $S(\mathfrak{q})^Q$, and that similar theorem holds for the invariants in the enveloping algebra. This is the most concise description of the algebra of invariants which we could hope for, and Kac asked whether this would hold for any algebraic Lie algebra provided p is sufficiently large [13]. A counterexample is given by the three dimensional solvable algebraic Lie algebra over $\mathbb{C}$ spanned by $\{h, a, b\}$ with nonzero brackets $[h, a] = a$ and $[h, b] = b$. In this case $S(\mathfrak{q})^Q = \mathbb{C}$ and so after reducing modulo p the element $a^{p-1}b$ of $S(\mathfrak{q}_p)$ is an example of an invariant for $\mathfrak{q}_p$ which is not generated in this way. Despite failing in general, we shall show that this nice behaviour holds for centralisers in type A and C, thus giving us a complete description of the symmetric invariant algebras in these cases.

Let us now introduce the notation required to state our first theorem. These notations shall be used henceforth without exception. Let $N \in \mathbb{N}$, let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and let V be an N-dimensional vector space. The group $G = GL(V)$ acts by conjugation on its Lie algebra $\mathfrak{g} = \mathfrak{gl}(V)$. Choose a non-degenerate bilinear form $(\cdot,\cdot)$ on V which is either symmetric or skew. We write $(u, v) = \epsilon(v, u)$ with $\epsilon = \pm 1$. The subgroup of G preserving the form shall be denoted K, and is either an orthogonal group or a symplectic group. Whenever we discuss such a group, we shall assume that $\mathrm{char}(\mathbb{K}) \neq 2$. The Lie algebra shall be denoted $\mathfrak{k}$ and is equal to the set of all $x \in \mathfrak{g}$ which are skew self-adjoint with respect to $(\cdot,\cdot)$. If we choose a basis for V then $(\cdot,\cdot)$ takes the form $(u, v) = u^\top J v$ where J is a matrix. There is a Lie algebra automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ of order 2 defined by

$$\sigma(X) = -J^{-1} X^\top J$$

which is independent of our choice of basis. Then $\mathfrak{k}$ coincides with the $+1$ eigenspace of σ. The -1 eigenspace shall be denoted $\mathfrak{p}$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{p}$ is a K-module.

If $x \in \mathfrak{g}$ then we may identify $\mathfrak{g}_x$ with $\mathrm{Lie}(G_x)$; see [10, Theorem 2.5]. The dual space to $\mathfrak{g}_x$ will be denoted $\mathfrak{g}_x^*$. If furthermore $x \in \mathfrak{k}$ then we may identify $\mathfrak{k}_x$ with $\mathrm{Lie}(K_x)$ for the same reason and, since $\sigma(x) = x$, the involution σ induces a decomposition $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{p}_x$ which is K_x-stable. The dual space $\mathfrak{k}_x^*$ identifies (as a K_x-module) with the annihilator of $\mathfrak{p}_x$ in $\mathfrak{g}_x$, whilst $\mathfrak{p}_x^*$ identifies with the annihilator of $\mathfrak{k}_x$ in $\mathfrak{g}_x^*$. By duality we identify the symmetric algebra $S(\mathfrak{g}_x)$ with $\mathbb{K}[\mathfrak{g}_x^*]$ as G_x-modules, and identify $S(\mathfrak{k}_x)$ and $S(\mathfrak{p}_x)$ with $\mathbb{K}[\mathfrak{k}_x^*]$ and $\mathbb{K}[\mathfrak{p}_x^*]$ as K_x-modules. Using these identifications we obtain restriction maps $S(\mathfrak{g}_x) \to S(\mathfrak{k}_x)$ and $S(\mathfrak{g}_x) \to S(\mathfrak{p}_x)$ which are K_x-module homomorphisms. Given a commutative algebra A we denote by A^p the pth power subalgebra $\{a^p\colon a \in A\}$.

Theorem 1. *Suppose that $\epsilon = -1$ so that K is of type* C. *If $Q \in \{G, K\}$ is of rank ℓ and $x \in \mathfrak{q} = \mathrm{Lie}(Q)$ then*

(1) $\mathbb{K}[\mathfrak{q}_x^*]^{\mathfrak{q}_x}$ *is free of rank p^ℓ over* $\mathbb{K}[\mathfrak{q}_x^*]^p$;
(2) $\mathbb{K}[\mathfrak{q}_x^*]^{Q_x}$ *is a polynomial algebra on ℓ generators*;
(3) $\mathbb{K}[\mathfrak{q}_x^*]^{\mathfrak{q}_x} \cong \mathbb{K}[\mathfrak{q}_x^*]^p \otimes_{(\mathbb{K}[\mathfrak{q}_x^*]^p)^{Q_x}} \mathbb{K}[\mathfrak{q}_x^*]^{Q_x}$.

It is easily seen that the above theorem has a straightforward reduction to the nilpotent case, for if $x = x_s + x_n$ is the Jordan decomposition of $x \in \mathfrak{q}$ then $\mathfrak{q}_x = (\mathfrak{q}_{x_s})_{x_n}$. But $\mathfrak{q}_{x_s}$ is a direct sum of the centre and of simple groups of types A or C, whilst the reductive rank of $\mathfrak{q}$ is equal to that of $\mathfrak{q}_{x_s}$. An inductive argument may then be used.

It will be convenient to discuss Lie algebras in some generality, and so we let $\mathfrak{q}$ be an arbitrary Lie algebra over $\mathbb{K}$ and let W be a $\mathfrak{q}$-module. For $\alpha \in W^*$ we define the stabiliser

$$\mathfrak{q}_\alpha = \bigl\{x \in \mathfrak{q}\colon \alpha(xw) = 0 \text{ for all } w \in W\bigr\}.$$