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Trees of fusion systems

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ABSTRACT

We define a 'tree of fusion systems' and give a sufficient condition for its completion to be saturated. We apply this result to enlarge an arbitrary fusion system by extending the automorphism groups of certain of its subgroups.

Saturated fusion systems have come into prominence during the course of the last two decades and may be viewed as a convenient language in which to study some types of algebraic objects at a particular prime. The original idea is due to Puig in [9], although our notation and terminology more closely follows that of Broto, Levi and Oliver in [2]. The area is now well established and has attracted wide interest from researchers in group theory, topology and representation theory. Formally, a fusion system on a finite p-group S is a category $\mathcal{F}$ whose objects consist of all subgroups of S and whose morphisms are certain group monomorphisms between objects. Saturation is an additional property, which is satisfied by many 'naturally occurring' fusion systems, including those induced by finite groups.

Recall that a tree of groups consists of a tree $\mathcal{T}$ with an assignment of groups to vertices and edges, and monomorphisms from 'edge groups' to incident 'vertex groups'. The completion of a tree of groups is the free product of all vertex groups modulo relations determined by the monomorphisms. The theory of trees of groups is a special case of Bass–Serre theory and this paper is an attempt to extend this theory to fusion systems (over a fixed prime p) by attaching fusion systems to vertices and edges of some fixed tree, and injective morphisms from edge to vertex fusion systems. As in the case for groups, the completion of a 'tree of fusion systems' is a colimit for the natural diagram, and we find a condition which renders this the fusion system generated by the vertex fusion systems.

One naturally obtains a tree of fusion systems from a tree of finite groups by replacing each group in the tree by its fusion systems at the prime p. Two questions arise at this point:

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(1) Is the completion of this tree of fusion systems the fusion system of the completion of the underlying tree of groups?
(2) Is it possible to prove that the completion is saturated 'fusion theoretically', i.e. without reference to the groups themselves?

We answer (1) in the affirmative essentially by applying a straightforward lemma of Robinson concerning conjugacy relations in completions of trees of groups:

Theorem A. *Let $(\mathcal{T},\mathcal{G})$ be a tree of finite groups and write $\mathcal{G}_{\mathcal{T}}$ for the completion of $(\mathcal{T},\mathcal{G})$. Let $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of fusion systems induced by $(\mathcal{T},\mathcal{G})$ which satisfies (H) so that there exists a completion $\mathcal{F}_{\mathcal{T}}$ for $(\mathcal{T},\mathcal{F},\mathcal{S})$. The following hold*:

(a) *$\mathcal{S}(v_*)$ is a Sylow p-subgroup of $\mathcal{G}_{\mathcal{T}}$.*
(b) *$\mathcal{F}_{\mathcal{S}(v_*)}(\mathcal{G}_{\mathcal{T}})=\mathcal{F}_{\mathcal{T}}$.*

In particular, $\mathcal{F}_{\mathcal{T}}$ is independent of the choice of tree of fusion systems $(\mathcal{T},\mathcal{F},\mathcal{S})$ induced by $(\mathcal{T},\mathcal{G})$.

In this theorem, (H) is a condition on $(\mathcal{T},\mathcal{F},\mathcal{S})$ which forces the existence of a completion $\mathcal{F}_{\mathcal{T}}$ on the p-group $\mathcal{S}(v_*)$ associated to a fixed vertex v_* of the underlying tree $\mathcal{T}$. To answer question (2) above, we require the introduction of some more terminology. Alperin's Theorem asserts that conjugacy in a saturated fusion system $\mathcal{F}$ on S is determined by the '$\mathcal{F}$-essential' subgroups and S. Conversely a deep result of Puig asserts that whenever conjugacy in $\mathcal{F}$ is determined by the '$\mathcal{F}$-centric' subgroups, $\mathcal{F}$ is saturated if the saturation axioms hold between such subgroups. We exploit both of these facts in the proof of the following theorem:

Theorem B. *Let $(\mathcal{T},\mathcal{F},\mathcal{S})$ be a tree of fusion systems which satisfies (H) and assume that $\mathcal{F}(v)$ is saturated for each vertex v of $\mathcal{T}$. Write $S:=\mathcal{S}(v_*)$ and $\mathcal{F}_{\mathcal{T}}$ for the completion of $(\mathcal{T},\mathcal{F},\mathcal{S})$. Assume that the following hold for each $P\leqslant S$*:

(a) *If P is $\mathcal{F}_{\mathcal{T}}$-conjugate to an $\mathcal{F}(v)$-essential subgroup or $P=\mathcal{S}(v)$ then P is $\mathcal{F}_{\mathcal{T}}$-centric.*
(b) *If P is $\mathcal{F}_{\mathcal{T}}$-centric then $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F})$ is a tree.*

Then $\mathcal{F}_{\mathcal{T}}$ is a saturated fusion system on S.

Theorem B in the case where $(\mathcal{T},\mathcal{F},\mathcal{S})$ is induced by a tree of groups $(\mathcal{T},\mathcal{G})$ is [3, Theorem 4.2], and can be deduced from Theorem B by applying Theorem A. The novelty of our approach is the introduction of the graph $\mathrm{Rep}_{\mathcal{F}_{\mathcal{T}}}(P,\mathcal{F})$ called the 'P-orbit graph' (defined for each $P\leqslant\mathcal{S}(v_*)$) which gives detailed information about the way in which P 'acts' on $\mathcal{F}_{\mathcal{T}}$. Condition (a) ensures that conjugacy in $\mathcal{F}_{\mathcal{T}}$ is determined by the $\mathcal{F}_{\mathcal{T}}$-centric subgroups and (b) ensures that the saturation axioms hold between such subgroups.

Theorem B is useful in determining fusion systems over specific (families of) p-groups. For example, in [8], Oliver applies [3, Theorem 5.1] (which follows from [3, Theorem 4.2]) to prove that saturated fusion systems over p-groups with abelian subgroup of index p are uniquely determined by their essential subgroups. Our next result is a generalisation of [3, Theorem 5.1] to arbitrary fusion systems.

Theorem C. *Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. For $1\leqslant i\leqslant m$, let $Q_i\leqslant S$ be a fully $\mathcal{F}_0$-normalised subgroup with $Q_i\varphi\nleqslant Q_j$ for each $\varphi\in\mathrm{Hom}_{\mathcal{F}_0}(Q_i,S)$ and $i\neq j$. Set $K_i:=\mathrm{Out}_{\mathcal{F}_0}(Q_i)$ and choose $\Delta_i\leqslant\mathrm{Out}(Q_i)$ so that K_i is a strongly p-embedded subgroup of Δ_i. Write*

$$\mathcal{F}=\left\langle\left\{\mathrm{Hom}_{\mathcal{F}_0}(P,S)\;\middle|\;P\leqslant S\right\}\cup\{\Delta_i\mid 1\leqslant i\leqslant m\}\right\rangle_S.$$

Assume further that for each $1\leqslant i\leqslant m$,