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Are we counting or measuring something? ☆

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ABSTRACT

Let H be a semisimple Hopf algebras over an algebraically closed field k of characteristic 0. We define Hopf algebraic analogues of commutators and their generalizations and show how they are related to H', the Hopf algebraic analogue of the commutator subgroup. We introduce a family of central elements of H', which on one hand generate H' and on the other hand give rise to a family of functionals on H. When $H = kG$, G a finite group, these functionals are counting functions on G. It is not clear yet to what extent they measure any specific invariant of the Hopf algebra. However, when H is quasitriangular they are at least characters on H.

Introduction

Commutators and commutator subgroups are some of the most fundamental concepts in group theory. These subgroups measure how far the group is from being commutative. It was Frobenius who proved early on that a function on a finite group G, that counts the number of ways an element of G can be realized as a commutator, is a character of G. This was done by giving an explicit formula for this counting function. Generalizations of this formula were proved throughout the years (e.g. [21,12,1]). The approaches varied from a direct approach through the use of a symmetric bilinear form and its associated Casimir element, to the use of distribution functions which are uniform on conjugacy classes of G.

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In this paper we define Hopf algebraic analogues of commutators and their generalizations and show how they are related to H', the Hopf algebraic analogue of the commutator subgroup. We introduce a family of elements in H' denoted by z_n, $n > 1$, which arise from the idempotent integral of H. This family consists of powers of the S-fixed central invertible element z_2,

$$z_2 = \sum \Lambda_1 \Lambda'_1 S\Lambda_2 S\Lambda'_2$$

where Λ, Λ' are two copies of the idempotent integral of H.

The elements $\{z_n\}$ are shown to give rise to Hopf algebraic analogues of the various counting functions for groups. On the other hand they are shown to be central Casimir elements associated to certain symmetric bilinear forms and Higman maps on H or on its center.

A different characterization of z_n is given when H is also assumed to be almost cocommutative. In this situation the z_n's are related to the so-called *generalized class sums* for H. Information about functionals related to iterated commutators can be deduced from the character table of H.

We use the following notations. Let H be a d-dimensional semisimple Hopf algebras over an algebraically closed field k of characteristic 0 with an idempotent integral Λ. Denote by $\{E_i\}_{0\leqslant i\leqslant n-1}$ the set of primitive central idempotents of H. Let d_i denote the degree of the irreducible character χ_i and let $R(H) = \mathrm{Sp}_k\{\chi_i\}$. Then $\chi_H = \lambda = \sum d_i\chi_i$ is an integral for H^*.

Let $\Psi : H_{H^*} \to H^*_{H^*}$ be the Frobenius map given by:

$$\Psi(h) = \lambda \leftharpoonup S(h).$$

The commutator subalgebra H' is a normal left coideal subalgebra of H for which $H/(HH'^{+})$ is commutative and it is minimal with respect to this property.

In Section 2 we define the commutator $\{a, b\}$ for $a, b \in H$ as follows:

$$\{a, b\} = \sum a_1 b_1 Sa_2 Sb_2.$$

We define also the general commutator $\{a^1, \ldots, a^n\}$, $a^i \in H$, as follows:

$$\{a^1, \ldots, a^n\} = \sum a^1_1 \cdots a^n_1 Sa^1_2 \cdots Sa^n_2.$$

We show that the general commutator can always be obtained as a sum of products of commutators.

Of special interest are commutators related to the idempotent integral Λ of H. Let Λ^i be a copy of the idempotent Λ. Set

$$z_n = \sum \Lambda^1_1 \cdots \Lambda^n_1 S\Lambda^1_2 \cdots S\Lambda^n_2, \qquad z_0 = 1.$$

We show:

Theorem 2.7: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then for all $k, n \geqslant 0$,

$$z_{2k+1} = z_{2k}, \qquad z_n = z_2^{\frac{n - n(\mathrm{mod}\ 2)}{2}} \in Z(H).$$

The commutator z_2 has a very nice form:

Theorem 2.8: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then

$$z_2 = \sum \Lambda^1_1 \Lambda^2_1 S\Lambda^1_2 S\Lambda^2_2 = \sum_i \frac{1}{d_i^2} E_i \in Z(H).$$