Journal of Algebra 398 (2014) 284–302

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Radical extensions for the Carlitz module

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ARTICLE INFO

Article history:
Received 17 July 2013
Available online 18 October 2013
Communicated by Eva Bayer-Fluckiger

MSC:
11R60
11R99
12F05
14H05

Keywords:
Carlitz module
Cogalois extension
Torsion module
Galois extension

ABSTRACT

Let L/K be a finite extension of congruence function fields. We say that L/K is a *radical extension* if L is generated by roots of polynomials $C_M(u)-\alpha \in K[u]$, where $C_M(u)$ is the action of Carlitz. We study a special class of these extensions, the *pure coradical* extensions. We prove that any pure coradical extension has order a power of the characteristic of K. We also give bounds for the Carlitz torsion of these extensions.

1. Introduction

Let L/K be a field extension. In the study of radical extensions we have the torsion and the cogalois classical groups

$$T(L/K) = \{u \in (L)^* \mid u^n \in K \text{ for some integer } n\}$$

and

$$\operatorname{cog}(L/K) = T(L/K)/(K)^*.$$

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.029

There exists an important class of field extensions studied by Greither and Harrison [4] which is called *cogalois extensions*. We say that L/K is a cogalois extension if:

(1) $L = K(T(L/K))$ and
(2) $\operatorname{card}(\operatorname{cog}(L/K)) \leqslant [L : K]$,

see [4] and [2].

The analogy of number fields with congruence function fields and, more precisely, of cyclotomic number fields with cyclotomic function fields, leads to the natural question if one can find an analog of the usual torsion group in the torsion module defined by the Carlitz action.

We give a new definition of radical extension by using the action of Carlitz. A function field extension L/K will be called *radical*, if L can be generated by some elements u with $C_{M_u}(u) \in K$ over K, where M_u are some polynomials. Among these extensions we study *pure coradical* extensions. An extension is called pure coradical, if it is radical, separable and *pure*. They can be viewed as the generalizations of Carlitz–Kummer extensions. These extensions have analogous properties to those of cogalois extensions defined in [4], see Sections 4 and 5. A pure coradical extension L/K satisfies $L = K(T(L/K))$. Observe the analogy with the previous definition.

In this paper we study the torsion given by the action of Carlitz. Thus we understand "radical" in the sense of this action. We study the structure of congruence function fields generated by torsion. In Section 3 we define the concept of pure coradical extension as an analogue of cogalois extensions in the classical case. We give examples of pure coradical and nonpure coradical extensions and of pure and nonpure extensions and show that, as in the classical case, the extension $k(\Lambda_{P^n})/k(\Lambda_P)$ is pure, where $P \in R_T = \mathbb{F}_q[T]$ is an irreducible polynomial and $n \in \mathbb{N}$. In Sections 4 and 5 we give some properties of radical and pure coradical extensions and prove, as in the classical case, that for Galois extensions, the cogalois group is isomorphic with the group of crossed homomorphisms.

In Section 6 we obtain our main results: we characterize the finite pure coradical extensions. In particular we prove that finite pure coradical extensions are p-extensions, where p is the characteristic of the base field. This is given in Theorems 6.6 and 6.7 and Corollary 6.9. Examples and applications are provided in Section 7. Finally, in Section 8 we find an upper bound for the cogalois group of a pure coradical extension.

In the classical case, Greither and Harrison [4] obtained that if L/K is a cogalois extension, then every intermediate field $K \subseteq F \subseteq L$ satisfies that F/K and L/F are cogalois. They also show the existence of a lattice isomorphism between intermediate extensions of L/K and subgroups of $\operatorname{cog}(L/K)$. Further, they proved that $|\operatorname{cog}(L/K)| = [L : K]$ and so on. None of these properties hold any more in our case, see Lemma 7.3, Examples 7.8, 7.10, 7.11, Proposition 8.4 and Example 8.7. Therefore we cannot expect a prefect analogy with the classical theory. For this reason, we do not call the extensions which we study "cogalois", the analogy is just not strong enough.

2. Notation

We shall use the following notation.

p denotes a prime number.
$q = p^\nu$, $\nu \in \mathbb{N}$.
$k = \mathbb{F}_q(T)$ denotes the field of rational functions.
$R_T = \mathbb{F}_q[T]$.
$\bar{k}$ denotes an algebraic closure of k.
$\operatorname{char}(L)$ denotes the characteristic of a field L.

If E/L is a field extension such that $k \subseteq L \subseteq E \subseteq \bar{k}$, we denote by $T(E/L)$ the set $\{u \in E \mid$ there exists $M \in R_T$ such that $C_M(u) \in L\}$.

C_m denotes the cyclic group of order m.