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On the structure of locally finite groups with small centralizers

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ABSTRACT

Let A stand for the group isomorphic with S_4, the symmetric group on four symbols. Let V denote the normal subgroup of order four in A and choose an involution $\alpha \in A \setminus V$. The Sylow 2-subgroup D of A is $V\langle\alpha\rangle$ and this is isomorphic with the dihedral group of order 8. We prove that if G is a locally finite group containing a subgroup isomorphic with D such that $C_G(V)$ is finite and $C_G(\alpha)$ has finite exponent, then $[G, D]'$ has finite exponent. If G is a locally finite group containing a subgroup isomorphic with A such that $C_G(V)$ is finite and $C_G(\alpha)$ has finite exponent, then G has finite exponent.

1. Introduction

A group is locally finite if every finite subset of the group generates a finite subgroup. In the theory of locally finite groups centralizers play an important role. In particular the following family of problems has attracted great deal of attention in the past. Let G be a locally finite group containing a finite subgroup V such that $C_G(V)$ is small in some sense. What can be said about the structure of G? In some situations quite a significant information about G can be deduced. For example if $|V| = 2$ and $C_G(V)$ is finite, then G has a nilpotent subgroup of class at most two with finite index bounded by a function of $|C_G(V)|$ by [3]. If G contains an element of prime order p whose centralizer is finite of order m, then G contains a nilpotent subgroup of finite (m, p)-bounded index and p-bounded nilpotency class. This result for locally nilpotent periodic groups is due to Khukhro [9], while the reduction to the nilpotent case was obtained combining a result of Hartley and Meixner [4] with that of Fong [1]. The latter uses the classification of finite simple groups. Another important result in this

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direction is Hartley's theorem that if G has an element of order n with finite centralizer of order m, then G contains a locally soluble subgroup with finite (m,n)-bounded index [6].

Infinite locally finite groups containing a non-cyclic subgroup with finite centralizer can be simple. One example is provided by the group $PSL(2,F)$, where F is an infinite locally finite field of odd characteristic. This group contains a non-cyclic subgroup of order four with finite centralizer. In the sequel we call the non-cyclic group of order four the four-group (or four-subgroup). In [16] the second author proved that if a locally finite group G contains a non-cyclic subgroup V of order p^2 for a prime p such that $C_G(V)$ is finite and $C_G(v)$ has finite exponent for all nontrivial elements $v \in V$, then G is almost locally soluble and has finite exponent. A group is said to almost have certain property if it contains a subgroup of finite index with that property. By a group of finite exponent we mean a group G for which there exists a positive integer e such that the order of any element of G divides e.

More recently new results on locally finite groups with small centralizer of a four-subgroup were proved [13,10]. In particular it was proved that if G is a locally finite group having a four-subgroup V such that $C_G(V)$ is finite and $C_G(v)$ has finite exponent for some $v \in V$, then $[G,v]'$ has finite exponent. This enables us to obtain a pretty detailed information on the structure of G (see [10] for more details). In the present paper we use results on automorphisms of finite groups [14] to deduce further theorems on the structure of locally finite groups with small centralizers.

We will now fix notation that will be used throughout the paper. Let A stand for the group isomorphic with S_4, the symmetric group on four symbols. Let V denote the normal subgroup of order four in A and choose an involution $\alpha \in A \setminus V$. The Sylow 2-subgroup D of A is $V\langle\alpha\rangle$ and this is isomorphic with the dihedral group of order 8. Choose an involution $\beta \in V$. So we have $V = \langle\beta, \beta^\alpha\rangle$ and $D = \langle\alpha, \beta\rangle$.

Theorem 1.1. *Let G be a locally finite group containing a subgroup isomorphic with D such that $C_G(V)$ is finite and $C_G(\alpha)$ has finite exponent. Then $[G,D]'$ has finite exponent.*

Since $[G,D]$ has finite index in G, the above theorem allows us to deduce some very specific information on the structure of G. Corollary 3.6 given in the end of the paper shows that under the hypothesis of Theorem 1.1 the group G has a normal series

$$1 \leqslant G_1 \leqslant G_2 \leqslant G_3 \leqslant G$$

such that G_1 has finite exponent, G/G_2 is finite and G_2/G_1 is abelian. Moreover G_3 is hyper-abelian and has finite index in G.

Theorem 1.2. *Let G be a locally finite group containing a subgroup isomorphic with A such that $C_G(V)$ is finite and $C_G(\alpha)$ has finite exponent. Then G has finite exponent.*

One important, if implicit, tool used in the proofs of our main theorems is the solution of the restricted Burnside problem [18,19]. In particular the results on automorphisms of finite groups [14] are based on the solution of the restricted Burnside problem.

Throughout the paper we use the expression "$(a,b,\ldots)$-bounded" to mean "bounded from above by some function depending only on $a,b,\ldots$".

2. Preliminaries

If B is a group of automorphisms of a group G, we will denote by $C_G(B)$ the set of all elements of G that are fixed by any automorphism from B. We denote by $[G,B]$ the subgroup generated by all elements of the form $g^{-1}g^b$, where $g \in G$ and $b \in B$. It is well-known that $[G,B]$ is a normal B-invariant subgroup of G. The first lemma is a collection of well-known facts on coprime automorphisms of finite groups. All proofs can be easily extracted for example from [2, 5.3.6, 6.2.2, 10.4.1].