Journal of Algebra 398 (2014) 314–317

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Products of elements of even order

George Glauberman

University of Chicago, 5734 S. University Ave., Chicago, IL, 60637, United States

ARTICLE INFO

Article history:
Received 8 February 2013
Available online 30 July 2013
Communicated by B. Külshammer

To Geoffrey Robinson on his 60th birthday

Keywords:
Involutions
Conjugate
Even order

ABSTRACT

The product of two non-conjugate elements of order two in a finite group must have even order. In this paper, we generalize this result for the product of two suitable elements of even order.

1. Introduction

In this article, we consider groups satisfying the following hypothesis:

(H) (1) G is a finite group;
  (2) u and v are involutions (elements of order two) in G;
  (3) t is a 2-element of G;
  (4) there are no conjugates u', v' of u, v such that $u'v' = t$.

(Recall that a 2-element is an element whose order is a nonnegative power of 2. Thus, the identity is a 2-element.)

For every subgroup H of G, let $O(H)$ be the largest normal subgroup of H of odd order. From the structure of a dihedral group, it is easy to see that under hypothesis (H), one can extend (4) to obtain:

(4′) there are no conjugates u', v' of u, v such that $u'v' = tw$ for some element w of odd order in $C_G(t)$.

Additional consequences of hypothesis (4) have been obtained by Richard Brauer in [2] using block theory. In this article, we use the methods of [2] to obtain the following further consequence:

E-mail address: gg@math.uchicago.edu.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.07.015

Theorem 1. *Assume hypothesis* (H)*. Suppose*

(a) *w is an element of odd order in $C_G(t)$;*
(b) *$u' \in O(C_G(u))$ and $v' \in O(C_G(v))$.*

Then $(uu')(vv') \neq tw$.

By taking the case when $t = 1$, we obtain:

Corollary. *Suppose u and v are involutions in a finite group G that are not conjugate in G. Assume that $u' \in O(C_G(u))$ and $v' \in O(C_G(v))$. Then $(uu')(vv')$ has even order.*

All groups in this article are finite. For a group G, we denote the set of irreducible complex characters of G by $\mathrm{Irr}(G)$ and the set of all elements of odd order in G by $G_{2'}$. For elements x, y, z of G, let

$$f(x, y, z) = \sum \chi(x)\chi(y)\overline{\chi(z)}/\chi(1)$$

where χ ranges over $\mathrm{Irr}(G)$.

Henceforth, we let G denote a fixed, but arbitrary, group.

2. Preliminary results

Let p be a prime. One may represent G by linear transformations of a vector space V over a splitting field of characteristic p. When G acts irreducibly on V, one obtains a complex-valued function on the set of p'-elements of G, called an *irreducible Brauer character*, as described in [6, pp. 16–18]. Now consider the union of $\mathrm{Irr}(G)$ with the set of all irreducible Brauer characters of G. By considering their values on group elements, we may partition the elements of this union into equivalence classes, called *p-blocks* [6, pp. 48–49] or simply *blocks*, if p is given. For a block B, we denote the set of irreducible complex characters in B by $\mathrm{Irr}(B)$. The class that contains the principal complex character of G is called the *principal p-block of G*.

Now we require some well-known preliminary results. The first follows from [5, Theorem 4.2.12].

Proposition 1 *(Class multiplication formula). Let x, y, z be elements of G and k be the number of ordered pairs (x', y') such that x' is conjugate to x in G; y' is conjugate to y in G; and $x'y' = z$. Then*

$$k = \left|G : C_G(x)\right|\left|G : C_G(y)\right| f(x, y, z)/|G|.$$

Henceforth, let $B_0(G)$ denote the principal 2-block of G.

The next two results are special cases of properties of p-blocks for all primes p. Part (b) of Proposition 2 ultimately derives from [3].

Proposition 2. *Let ϕ be an irreducible Brauer character of G for the prime* 2*.*

(a) *If ϕ lies in $B_0(G)$, then $\phi(x) = \phi(1)$ for every element x of $O(G)$.*
(b) *If ϕ lies outside $B_0(G)$, then*

$$\sum \phi(x) = 0,$$

where the sum is taken over $G_{2'}$.