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Torsion-free endotrivial modules

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ABSTRACT

Let G be a finite group and let $T(G)$ be the abelian group of equivalence classes of endotrivial kG-modules, where k is an algebraically closed field of characteristic p. We investigate the torsion-free part $TF(G)$ of the group $T(G)$ and look for generators of $TF(G)$. We describe three methods for obtaining generators. Each of them only gives partial answers to the question but we obtain more precise results in some specific cases. We also conjecture that $TF(G)$ can be generated by modules belonging to the principal block and we prove the conjecture in some cases.

1. Introduction

If G is a finite group, let $T(G)$ be the group of equivalence classes of endotrivial kG-modules, where k is a field of characteristic p (assumed algebraically closed for simplicity). The abelian group $T(G)$ is finitely generated, hence of the form $T(G) = TT(G) \oplus F$, where $TT(G)$ is the torsion subgroup and F is a free abelian group. The purpose of this paper is to investigate the torsion-free part of $T(G)$, and in particular find generators for a suitable torsion-free direct summand F of $T(G)$. The non-uniqueness of F is actually an issue and so we work instead with the canonically defined free abelian group $TF(G) = T(G)/TT(G)$.

In many cases, $TF(G) \cong \mathbb{Z}$, generated by the class of the syzygy module $\Omega(k)$. Otherwise, by [9], G has maximal elementary abelian p-subgroups of rank 2 and its Sylow p-subgroup P has a rather special structure. In particular, the center $Z(P)$ is cyclic, hence has a unique subgroup Z of order p.

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In order to find generators for $TF(G)$, there are three known constructions, one using relative syzygies, one using suitable subquotients of a syzygy module $\Omega^n(k)$, and one involving a class in group cohomology restricting non-trivially to Z. We analyze the three constructions and extend as much as possible the results about them. The first construction works well for p-groups and also for groups with a normal Sylow p-subgroup, but it does not seem possible to extend the method to arbitrary finite groups. The second construction needs the assumption that Z is normal in G and hence cannot work otherwise. The third construction, which we call the cohomological pushout method, works well in general, but only rationally, not integrally: it provides generators for $\mathbb{Q} \otimes_{\mathbb{Z}} TF(G)$, but it produces only a subgroup of finite index in $TF(G)$. We can show that this subgroup is the whole of $TF(G)$ in some cases, but we also give examples where this is not so. The problem of describing generators of $TF(G)$ in full generality remains open, but our discussion shows where the difficulties lie and allows us to state specific questions to be solved.

We then prove that the endotrivial modules in the principal block form a subgroup $T_0(G)$ of $T(G)$ and that $T_0(G)$ has finite index. We conjecture that $T(G) = T_0(G) + TT(G)$, in other words that $TF(G)$ can be generated by modules in the principal block. We prove that the conjecture holds in some cases, in particular if Z is a normal subgroup.

Finally we discuss control of p-fusion and we conjecture that if P is a common Sylow p-subgroup of G and G' and if a group homomorphism $\phi : G \to G'$ induces an isomorphism $\mathcal{F}_P(G) \to \mathcal{F}_P(G')$ between the canonical fusion systems of G and G' on P, then ϕ induces an isomorphism $TF(G') \to TF(G)$. We prove the conjecture in a few cases.

In a final section, we have gathered a number of examples illustrating various results of this paper.

A general remark about our methods may be useful. If $N_G(P)$ denotes the normalizer of a Sylow p-subgroup P of G, many results can be proved for $N_G(P)$ but the passage from $N_G(P)$ to G seems difficult. It is known that the restriction map $\mathrm{Res}^G_{N_G(P)} : T(G) \longrightarrow T(N_G(P))$ is injective, induced by the Green correspondence, but the non-surjectivity of this map is a crucial issue and is an obstacle for solving several of our problems (see Section 8).

2. Preliminaries

Throughout this paper, we let k denote an algebraically closed field of prime characteristic p. In addition, we assume that all modules are finitely generated.

Given a finite group H, we write k for the trivial kH-module, or, whenever H needs to be clarified, we write k_H instead. Unless otherwise specified, the symbol $\otimes$ is the tensor product $\otimes_k$ of the underlying vector spaces, and in case of kH-modules, then H acts diagonally on the factors. If M is a kH-module, and $\varphi : Q \to M$ its projective cover, then we let $\Omega^1(M)$, or simply $\Omega(M)$, denote the kernel of φ (called the first syzygy of M). Likewise, if $\vartheta : M \to Q$ is the injective hull of M (recall that kH is a self-injective ring so Q is also projective), then $\Omega^{-1}(M)$ denotes the cokernel of ϑ. Inductively, we set $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for all integers $n > 1$.

If G is a finite group of order divisible by p, then a kG-module M is *endotrivial* if its endomorphism algebra $\mathrm{End}_k(M)$ is isomorphic (as a kG-module) to the direct sum of the trivial module k_G and a projective kG-module. In other words, a kG-module M is endotrivial if and only if $M^* \otimes M \cong k \oplus (\mathrm{proj})$, where M^* denotes the k-dual $\mathrm{Hom}_k(M, k)$ of M, and (proj) some projective module. Recall the following basic results (see Section 2 in [9]).

Lemma 2.1. *Let G be a finite group of order divisible by p.*

(1) *Let M be a kG-module. If M is endotrivial, then M splits as the direct sum $M_\diamond \oplus (\mathrm{proj})$ for an indecomposable endotrivial kG-module $M_\diamond$, which is unique up to isomorphism.*
(2) *The relation*

$$M \sim N \quad \Longleftrightarrow \quad M_\diamond \cong N_\diamond$$

on the class of endotrivial kG-modules is an equivalence relation. We let $T(G)$ be the set of equivalence classes. Every equivalence class contains a unique indecomposable module up to isomorphism.