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Regular and p-regular orbits of solvable linear groups ☆

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ABSTRACT

Let V be a faithful G-module for a finite group G and let p be a prime dividing $|G|$. An orbit v^G for the action of G on V is p-regular if $|v^G|_p = |G : \mathbf{C}_G(v)|_p = |G|_p$. Zhang asks the following question in Zhang (1993) [8]. Assume that a finite solvable group G acts faithfully and irreducibly on a vector space V over a finite field $\mathbb{F}$. If G has a p-regular orbit for every prime p dividing $|G|$, is it true that G will have a regular orbit on V? In Lü and Cao (2000) [4], Lü and Cao construct an example showing that the answer to this question is no, however the example itself is not correct. In this paper, we study Zhang's question in detail. We construct examples showing that the answer to this question is no in general. We also prove the following result. Assume a finite solvable group G of odd order acts faithfully and irreducibly on a vector space V over a field of odd characteristic. If G has a p-regular orbit for every prime p dividing $|G|$, then G will have a regular orbit on V.

1. Introduction

Let V be a faithful G-module for a finite group G and let p be a prime dividing $|G|$. An orbit v^G for the action of G on V is p-regular if $|v^G|_p = |G : \mathbf{C}_G(v)|_p = |G|_p$. Zhang asks the following question in [8]. Assume that a finite solvable group G acts faithfully and irreducibly on a vector space V over a finite field $\mathbb{F}$. If G has a p-regular orbit for every prime p dividing $|G|$, is it true that G will have a regular orbit on V? In [4], Lü and Cao construct an example showing that the answer to this question

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is no. However the example itself is not correct. We mention that Lewis is the first to observe that the example in [4] is wrong in his review [5]. In this paper, we study Zhang's question in detail.

First we construct examples showing that the answer to this question is no in general.

(1) Example 1: Let $H \cong Z_3$ act faithfully and irreducibly on $V_1 = \mathbb{F}_2^2$. Then $G \cong H \wr Z_5$ acts on $V = \mathbb{F}_2^{10}$. Clearly G acts faithfully and irreducibly on V. The group G has a 3-regular orbit and a 5-regular orbit, but it has no regular orbit on V.
(2) Example 2: Let H be a solvable group acts faithfully, irreducibly and primitively on $\mathbb{F}_7^2$ and $H \cong Q_8 \rtimes S_3$, $\mathbf{Z}(H) \cong Z_2$. We may view H to be a matrix group and define G to be the central product of H with H, i.e. $G \cong H \curlyvee H$. Thus G acts faithfully, irreducibly and primitively on $V = \mathbb{F}_7^4$. $|G| = 1152 = 2^7 \cdot 3^2$. By direct calculation of GAP [2], the lengths of all the orbits of G on V are given in the following list:

$$(1, 48, 48, 48, 144, 144, 144, 192, 192, 192, 288, 288, 288, 384).$$

From the list we know that G has a 2-regular orbit and a 3-regular orbit, but it has no regular orbit on V.

In Example 1, G is a group of odd order induced from a group of odd order acting on a vector space over a field of characteristic 2. In Example 2, G is a solvable group of even order acting on a vector space over a field of odd characteristic. Based on these examples, it is natural to ask the following question. Assume that a finite solvable group G of odd order acts faithfully and irreducibly on a vector space V over a field of odd characteristic. If G has a p-regular orbit for every prime p dividing $|G|$, is it true that G will have a regular orbit on V? We answer this question affirmatively in the main theorem of this paper. We prove the following.

Theorem 1.1. *Assume that a finite solvable group G of odd order acts faithfully and irreducibly on a vector space V over a field of odd characteristic. If G has a p-regular orbit for every prime p dividing $|G|$, then G will have a regular orbit on V.*

2. Notation and lemmas

Before we prove the main result, we extract some important information from the work of Turull [7, Section 1] in the following propositions. Note that Turull's results are stated for a prime p but the same arguments will work for replacing p with a prime power q. We include the proof here for completeness.

Notation: Let q be a prime power and n an integer.

(1) $F(q^n) = \mathrm{GF}(q^n)^\times$ (the multiplicative group of the field of q^n elements).
(2) $\mathrm{Gal}(q^n) = \mathrm{Gal}(\mathrm{GF}(q^n) : \mathrm{GF}(q))$.
(3) $G(q^n) = \mathrm{Gal}(q^n) \ltimes F(q^n)$.
(4) If $\sigma \in \mathrm{Gal}(q^n)$ and $y \in F(q^n)$ we denote $N_\sigma(y) = \prod_{\tau \in \langle \sigma \rangle} \tau(y)$.
(5) Suppose that s is a prime and divides $n = |\mathrm{Gal}(q^n)|$. Define $\mathrm{GN}(q^n, s) = A \ltimes N \subseteq G(q^n)$ where A is a subgroup of $\mathrm{Gal}(q^n)$ of order s and $N = \{x \in F(q^n)\colon \prod_{\sigma \in A} \sigma(x) = 1\}$.

Proposition 2.1. a) *Let $\sigma \in \mathrm{Gal}(q^n)$ be of order s and set $N = \{x \in F(q^n)\colon N_\sigma(x) = 1\}$. Then we have that $x \in N$ iff $x = \frac{\sigma(y)}{y}$ for some $y \in F(q^n)$. Furthermore we have*

$$|N| = \frac{q^n - 1}{q^{n/s} - 1}.$$