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Self-similar associative algebras

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ABSTRACT

Famous self-similar groups were constructed by Grigorchuk, Gupta and Sidki, these examples lead to interesting examples of associative algebras. The authors suggested examples of self-similar Lie algebras in terms of differential operators. Recently Sidki introduced an example of an associative algebra of self-similar matrices. We construct families of self-similar associative algebras $\mathbf{C}_\Omega$, $\mathbf{D}_\Omega$, generalizing the example of Sidki. We prove that our algebras are $\mathbb{Z}\oplus\mathbb{Z}$-graded and have polynomial growth. Our approach is the weight strategy developed by the authors for self-similar Lie algebras and their envelopes. In particular, we obtain similar *triangular decompositions* into direct sums of three subalgebras $\mathbf{C}=\mathbf{C}_+\oplus\mathbf{C}_0\oplus\mathbf{C}_-$, $\mathbf{D}=\mathbf{D}_+\oplus\mathbf{D}_0\oplus\mathbf{D}_-$. We prove that some of our algebras are direct sums of two locally nilpotent subalgebras $\mathbf{C}=\mathbf{C}_+\oplus\mathbf{C}_-$, $\mathbf{D}=\mathbf{D}_+\oplus\mathbf{D}_-$. We show that in some cases the zero components $\mathbf{C}_0$, $\mathbf{D}_0$ are nontrivial and not nil algebras.
We show that our construction includes the example of Sidki and the examples of self-similar Lie algebras and their associative hulls constructed by the authors before.

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1. Introduction

The famous finitely generated periodic group was constructed by Grigorchuk [7]. Similar groups were constructed by Gupta and Sidki [10], their groups act on trees. All these groups are self-similar, we refer the reader for further details and developments to [8,9]. The study of these groups lead to investigation of group rings and other related associative algebras [22]. In particular, there appeared self-similar associative algebras defined by matrices in a recurrent way [4,5]. The recurrence matrices were also introduced in order to study number theoretical problems [1,2].

Let X be a set in an associative algebra, then by $\mathrm{Alg}(X)$ denote the associative subalgebra generated by X, similarly by $\mathrm{Lie}(X)$ (or $\mathrm{Lie}_p(X)$) denote the (restricted) Lie subalgebra generated by X. For necessary definitions and properties of the restricted Lie algebras we refer the reader to [11,24,3].

In [17,21] we introduced a family of self-similar Lie algebras $\mathbf{L}$ over fields of prime characteristic $p>0$ whose properties resemble those of Grigorchuk and Gupta–Sidki groups. The restricted Lie algebra $\mathbf{L}$ is generated by the following two derivations of the truncated polynomial ring $K[t_i \mid i \geqslant 0]/(t_i^p \mid i \geqslant 0)$:

$$v_1 = \partial_1 + t_0^{p-1}\bigl(\partial_2 + t_1^{p-1}\bigl(\partial_3 + t_2^{p-1}\bigl(\partial_4 + t_3^{p-1}\bigl(\partial_5 + t_4^{p-1}(\partial_6 + \cdots)\bigr)\bigr)\bigr)\bigr),$$
$$v_2 = \partial_2 + t_1^{p-1}\bigl(\partial_3 + t_2^{p-1}\bigl(\partial_4 + t_3^{p-1}\bigl(\partial_5 + t_4^{p-1}(\partial_6 + \cdots)\bigr)\bigr)\bigr).$$

We also studied the properties of the associative hull of these operators $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$ and the restricted enveloping algebra $u(\mathbf{L})$ [18,20]. Krylyouk proved that in so-called rational case the algebra $\mathbf{A}$ is not nil [15].

In our constructions we were motivated by analogies with constructions of self-similar groups. In particular, the following property is analogous to the *periodicity* of the Grigorchuk and Gupta–Sidki groups [7,10].

Theorem 1.1. *(See [17,21].) Let $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ be the restricted Lie algebra generated by $\{v_1, v_2\}$. Then $\mathbf{L}$ has a nil p-mapping.*

The second example from the recent work of Sidki [23] is the algebra $R_2 = \mathrm{Alg}(s_1, r_1)$ of self-similar matrices given by recursion as follows

$$s_i = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}, \qquad r_i = \begin{pmatrix} 0 & r_{i+1} \\ 0 & s_{i+1} \end{pmatrix}; \quad i \geqslant 1. \tag{1}$$

In particular, he proved that this ring has polynomial growth and that the semigroup generated by generators $\{s_1, r_1\}$ consists of nil elements [23].

In the present paper we do not develop a general theory of self-similar associative algebras, in particular we do not define the notion. We are rather concentrated on particular examples and their properties.

We start with the definition of the space W (Section 2) and introduce the elements s_i, r_i, $i \geqslant 1$, acting on it (Section 3). Next, we suggest families of self-similar associative algebras $\mathbf{C}_\Omega, \mathbf{D}_\Omega$ (Section 4) depending on array of parameters Ω. We define weight functions (Section 5) similar to our approach to self-similar Lie algebras [18,20,21]. In particular, we establish that these algebras are $\mathbb{Z} \oplus \mathbb{Z}$-graded (Theorem 5.2). Elements of our algebras are expressed via standard monomials and we obtain bounds on their weights (Section 6). We use these bounds to prove that our algebras have polynomial growth (Theorem 7.1). In Section 8 we establish *triangular decompositions* of our algebras into direct sums of three subalgebras $\mathbf{C} = \mathbf{C}_+ \oplus \mathbf{C}_0 \oplus \mathbf{C}_-$, $\mathbf{D} = \mathbf{D}_+ \oplus \mathbf{D}_0 \oplus \mathbf{D}_-$. We prove that some of our algebras are direct sums of two locally nilpotent subalgebras $\mathbf{C} = \mathbf{C}_+ \oplus \mathbf{C}_-$, $\mathbf{D} = \mathbf{D}_+ \oplus \mathbf{D}_-$ (Theorem 8.3). In some cases the zero components $\mathbf{C}_0$, $\mathbf{D}_0$ are nontrivial and not nil algebras (Theorem 9.1), this is a generalization of the result of Krylyouk [15].