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On the topology of relative and geometric orbits for actions of algebraic groups over complete fields

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ABSTRACT

In this paper, we investigate the problem of closedness of (relative) orbits for the action of algebraic groups on affine varieties defined over complete fields in its relation with the problem of equipping a topology on cohomology groups (sets) and give some applications.

Introduction

Let G be a smooth affine (i.e. linear) algebraic group acting regularly on an affine variety X, all are defined over a field k. Many results of (geometric) invariant theory related to the orbits of the action of G are obtained in the geometric case, i.e., when k is an algebraically closed field. However, since the very beginning of modern geometric invariant theory, as presented in [25,26], there is a need to consider the relative case of the theory. For example, Mumford has considered many aspects of the theory already over sufficiently general base schemes, with arithmetical aim (say, to construct arithmetic moduli of abelian varieties, as in Chapter 3 of [25,26]). Also some questions or conjectures due to Borel [8], Tits [25], etc. ask for extensions of results obtained to the case of non-algebraically closed fields. As typical examples, we just cite the results by Birkes [7], Kempf [18], Raghunathan [28], etc. to name a few, which gave the solutions to some of the above mentioned questions or conjectures.

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Besides, due to the need of number-theoretic applications, the local and global fields k are in the center of such investigation. For example, let an algebraic k-group G act on a k-variety V, $x \in V(k)$. We are interested in the set $G(k).x$, which is called *relative orbit of x* (to distinguish with *geometric orbit* $G.x$). One of the main steps in the proof of the analog of Margulis' super-rigidity theorem in the global function field case (see [40,19,20]) was to prove the (locally) closedness of some relative orbits $G(k).x$, $x \in V(k)$, for some action of an almost simple simply connected group G on a k-variety V. Moreover, when one considers some arithmetical rings (say, the integers ring, or the adèle ring of a global field) instead of k, this leads to Arithmetic Invariant Theory (see [5,6]) which plays an important role in the current study of arithmetic of elliptic and related curves over global fields. In this paper we assume that k is a field, complete with respect to a non-trivial valuation v of real rank 1 (e.g. $\mathfrak{p}$-adic field or the field of real numbers $\mathbf{R}$, i.e., a local field). Then for any affine k-variety X, we can endow $X(k)$ with the (Hausdorff) v-adic topology induced from that of k. Let $x \in X(k)$ be a k-point. We are interested in a connection between the Zariski-closedness of the orbit $G.x$ of x in X, and Hausdorff closedness of the relative orbit $G(k).x$ of x in $X(k)$. The first result of this type was obtained by Borel and Harish-Chandra [10] and then by Birkes [7], see also Slodowy [36] in the case $k = \mathbf{R}$, the real field, and then by Bremigan (see [14]). In fact, it was shown that if G is a reductive $\mathbf{R}$-group, $G.x$ is Zariski closed if and only if $G(\mathbf{R}).x$ is closed in the real topology (see [7,36]), and this was extended to $\mathfrak{p}$-adic fields in [14]. Notice that some of the proofs previously obtained in [7,14], etc. do not extend to the case of positive characteristic. The aim of this note is to see to what extent the above results still hold for more general class of algebraic groups and complete fields. In the course of study, it turns out that this question has a close relation with the problem of equipping a topology on cohomology groups (or sets), which has important aspects, say in duality theory for Galois or flat cohomology of algebraic groups in general (see [30,22,34,35]). We emphasize that, in the case char.$k = p > 0$, the stabilizer of a (closed) point needs not be a smooth subgroup, and the treatment of smoothness condition plays an important role here. The most satisfactory results are obtained for perfect fields, and also for a general class of groups over local fields. We have the following general results regarding some relations between the topology of relative orbits and that of geometric orbits.

Theorem 1. *Let k be a field, complete with respect to a real valuation of rank 1, G a smooth affine k-group, acting k-regularly on an affine k-variety V, $v \in V(k)$, and G_v the stabilizer of v in G.*

(1) (a) *The relative orbit $G(k).v$ is Hausdorff closed in $(G.v)(k)$. Thus if $G.v$ is Zariski closed in V, then $G(k).v$ is Hausdorff closed in $V(k)$.*
   (b) *(See [11,12,14].) If moreover, the stabilizer G_v of v is smooth over k, then for any $w \in (G.v)(k)$, the relative orbit $G(k).w$ is open and closed in Hausdorff topology of $(G.v)(k)$.*
(2) *Assume that $G(k).v$ is Hausdorff closed in $V(k)$. Then if either*
   (a) *G is nilpotent, or*
   (b) *G is reductive and the action of G is strongly separable,*
   then $G.v$ is Zariski closed in V. Therefore, in these cases, $G.v$ is Zariski closed in V if and only if $G(k).v$ is Hausdorff closed in $V(k)$.
(3) *Assume further that k is a perfect field, $G = L \times_k U$, where L is a reductive and U is a unipotent subgroup of G, L is defined over k, V, v are as above. Then $G(k).v$ is Hausdorff closed in $V(k)$ if and only if $G.v$ is Zariski closed in V.*

Here the action of G is said to be *strongly separable* (after [29]) at v if for all $x \in \mathrm{Cl}(G.v)$, the stabilizer G_x is smooth, or equivalently, the induced morphism $G \to G/G_x$ is separable.

One of the main tools to prove the theorem is the introduction of some specific topologies on the (Galois or flat) group cohomology and their relation with the problem of detecting the closedness of a given relative orbit. The main ingredient is the following theorem proved in [4], where we refer to Section 1 for the notion of special and canonical topology on the cohomology set $\mathrm{H}^1_{flat}(k, G)$.

Theorem 2. *(See [4].) Let G be an affine group scheme of finite type defined over a field k, complete with respect to a valuation of real rank 1. Then*