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Hilbert depth of graded modules over polynomial rings in two variables

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ABSTRACT

In this article we mainly consider the positively $\mathbb{Z}$-graded polynomial ring $R = \mathbb{F}[X, Y]$ over an arbitrary field $\mathbb{F}$ and Hilbert series of finitely generated graded R-modules. The central result is an arithmetic criterion for such a series to be the Hilbert series of some R-module of positive depth. In the generic case, that is $\deg(X)$ and $\deg(Y)$ being coprime, this criterion can be formulated in terms of the numerical semigroup generated by those degrees.

1. Introduction and review

We want to investigate how some of the results of [1] for Hilbert series of finitely generated graded modules over the standard $\mathbb{Z}$-graded polynomial ring can be generalised to the case where the ring of polynomials is endowed with an arbitrary positive $\mathbb{Z}$-grading.

Let $R = \mathbb{F}[X_1, \ldots, X_n]$ be the positively $\mathbb{Z}$-graded polynomial ring over some field $\mathbb{F}$, i.e. each X_i has degree $d_i \geqslant 1$ for every $i = 1, \ldots, n$. Moreover, let M be a finitely generated graded R-module.

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Every homogeneous component of M is a finite-dimensional $\mathbb{F}$-vector space, and since R is positively graded and M is finitely generated, $M_j = 0$ for $j \ll 0$. Hence the *Hilbert function* of M

$$H(M, -) : \mathbb{Z} \to \mathbb{Z}, \quad j \mapsto \dim_{\mathbb{F}}(M_j),$$

is a well-defined *integer Laurent function* (see [2, Definitions 5.1.1 and 5.1.12]). The formal Laurent series *associated to* $H(M, -)$

$$H_M(t) = \sum_{j\in\mathbb{Z}} H(M, j)t^j = \sum_{j\in\mathbb{Z}} (\dim_{\mathbb{F}} M_j)t^j \in \mathbb{Z}[[t]][t^{-1}]$$

is called the *Hilbert series* of M. Obviously it has no negative coefficients; such a series will be called *nonnegative* for short.

By the theorem of Hilbert–Serre (see [3, Thm. 4.1.1]), H_M may be written as a fraction of the form

$$\frac{Q_M(t)}{\prod_{i=1}^n (1 - t^{d_i})},$$

with some $Q_M \in \mathbb{Z}[t, t^{-1}]$. As a consequence of this theorem and a well-known result in the theory of generating functions, see Proposition 4.4.1 of [4], there exists a quasi-polynomial P of period $d := \mathrm{lcm}(d_1, \dots, d_n)$ such that $\dim_{\mathbb{F}}(M_j) = P(j)$ for $j \gg 0$.

The ring R is *local, that is, it has a unique maximal graded ideal, namely $\mathfrak{m} := (X_1, \dots, X_n)$. The *depth* of M is defined as the maximal length of an M-regular sequence in $\mathfrak{m}$, i.e. the grade of $\mathfrak{m}$ on M, and denoted by $\operatorname{depth}(M)$ rather than $\operatorname{grade}(\mathfrak{m}, M)$. This deviation from the standard terminology, where "depth" is used exclusively in the context of true local rings, may be justified by the fact that $\operatorname{grade}(\mathfrak{m}, M)$ agrees with $\operatorname{depth}(M_{\mathfrak{m}})$, see [5, Prop. 1.5.15].

It is easy to see that (contrary to the Krull dimension) the depth of a module M is not encoded in its Hilbert series. Therefore it makes sense to introduce

$$\operatorname{Hdep}(M) := \max\left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there is a f. g. gr. } R\text{-module } N \\ \text{with } H_N = H_M \text{ and } \operatorname{depth}(N) = r \end{array} \right\};$$

this number is called the *Hilbert depth* of M.

If the ring R is standard graded, then $\operatorname{Hdep}(M)$ turns out to coincide with the arithmetical invariant

$$\mathrm{p}(M) := \max\{ r \in \mathbb{N} \mid (1 - t)^r H_M(t) \text{ is nonnegative} \},$$

called the *positivity* of M, see Theorem 3.2 of [1]. The inequality $\operatorname{Hdep}(M) \leqslant \mathrm{p}(M)$ follows from general results on Hilbert series and regular sequences. The converse can be deduced from the main result of [1], Theorem 2.1, which states the existence of a representation

$$H_M(t) = \sum_{j=0}^{\dim(M)} \frac{Q_j(t)}{(1 - t)^j} \quad \text{with nonnegative } Q_j \in \mathbb{Z}[t, t^{-1}].$$

We begin our investigation by establishing a similar decomposition theorem for Hilbert series of modules over any positively $\mathbb{Z}$-graded polynomial ring. This result has some consequences for the Hilbert depth, but it does not lead to an analogue of the equation $\operatorname{Hdep}(M) = \mathrm{p}(M)$ – the occurrence of different factors in the denominator of H_M complicating matters. In Section 3 we restrict our attention to polynomial rings in two variables. For this special case we deduce an arithmetic characterisation of positive Hilbert depth. This criterion, surprisingly related to the theory of *numerical semigroups*, is the main result of our paper.