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Liftings and quasi-liftings of DG modules

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ABSTRACT

We prove lifting results for DG modules that are akin to Auslander, Ding, and Solberg's famous lifting results for modules.

Introduction

Convention. Throughout this paper, let R be a commutative noetherian ring.

Hochster famously wrote that "life is really worth living" in a Cohen–Macaulay ring [7].[2] For instance, if R is Cohen–Macaulay and local with maximal regular sequence $\underline{t}$, then $R/(\underline{t})$ is artinian and

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[2] We know of this quote from [5].

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the natural epimorphism $R \to R/(\underline{t})$ is nice enough to allow for transfer of properties between the two rings. Thus, if one can prove a result for artinian local rings, then one can (often) prove a similar result for Cohen–Macaulay local rings by showing that the desired conclusion descends from $R/(\underline{t})$ to R. When R is complete, then this is aided sometimes by the lifting result of Auslander, Ding, and Solberg [2, Propositions 1.7 and 2.6].

Theorem. *Let $\underline{t} \in R$ be an R-regular sequence, and let M be a finitely generated $R/(\underline{t})$-module. Assume that R is local and $(\underline{t})$-adically complete.*

(a) *If $\mathrm{Ext}^2_{R/(\underline{t})}(M,M)=0$, then M is "liftable" to R, that is, there is a finitely generated R-module N such that $R/(\underline{t}) \otimes_R N \cong M$ and $\mathrm{Tor}_i^R(R/(\underline{t}),N)=0$ for all $i \geqslant 1$.*
(b) *If $\mathrm{Ext}^1_{R/(\underline{t})}(M,M)=0$, then M has at most one lift to R.*

In this paper, we are concerned with what happens when the sequence $\underline{t}$ is not R-regular. One would like a similar mechanism for reducing questions about arbitrary local rings to the artinian case.

It is well known that the map $R \to R/(\underline{t})$ is not nice enough in general to guarantee good descent/lifting behavior. Our perspective[3] in this matter is that this is not the right map to consider in general: the correct one is the natural map from R to the Koszul complex $K = K^R(\underline{t})$. This perspective requires one to make some adjustments. For instance, K is a differential graded R-algebra, so not a commutative ring in the traditional sense. This may cause some consternation, but the payoff can be handsome. For instance, in [8] we use this perspective to answer a question of Vasconcelos [9]. One of the tools for the proof of this result is the following version of Auslander, Ding, and Solberg's lifting result. Note that we do not assume that R is local in part (a) of this result.

Main Theorem. *Let $\underline{t} = t_1, \dots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite.*

(a) *If $\mathrm{Ext}^2_{K^R(\underline{t})}(D,D)=0$, then D is quasi-liftable to R, that is, there is a semi-free R-complex D' such that $D \simeq K^R(\underline{t}) \otimes_R D'$.*
(b) *Assume that R is local. If D is quasi-liftable to R and $\mathrm{Ext}^1_{K^R(\underline{t})}(D,D)=0$, then any two homologically degreewise finite quasi-lifts of D to R are quasiisomorphic over R.*

This result is proved in Corollaries 3.7 and 3.12, which follow from more general results on liftings along morphisms of DG algebras. Note that it is similar to, but quite different from, some results of Yoshino [10].

We briefly describe the contents of the paper. Section 1 contains some background material on DG algebras and DG modules. Section 2 contains some structural results for DG modules and homomorphisms between them. Finally, Section 3 is where we prove our Main Theorem.

1. DG modules

We assume that the reader is familiar with the category of R-complexes. For clarity, we include a few definitions.

Definition 1.1. In this paper, complexes of R-modules ("R-complexes" for short) are indexed homologically:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots.$$

[3] This perspective is not original to our work. We learned of it from Avramov and Iyengar.