Journal of Algebra 373 (2013) 207–222

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Three dimensional canonical singularities in codimension two in positive characteristic

Masayuki Hirokado [a,*], Hiroyuki Ito [b], Natsuo Saito [a]

[a] Graduate School of Information Sciences, Hiroshima City University, 3-4-1 Ozuka-higashi, Asaminami-ku, Hiroshima 731-3194, Japan
[b] Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba 278-8510, Japan

ARTICLE INFO

Article history:
Received 3 February 2011
Available online 25 October 2012
Communicated by Steven Dale Cutkosky

Keywords:
Canonical singularities
Rational singularities
Deformations of isolated singularities
Grauert–Riemenschneider vanishing theorem

ABSTRACT

We investigate local structure of a three dimensional variety X defined over an algebraically closed field k of characteristic $p > 0$ with at most canonical singularities. Under the assumption that $p \geqslant 3$ and a general hyperplane cut of X has at most rational singularities, we show that local structure of X in codimension two is well understood in the level of local equations. Consequently, we find that i) any singularity of such a variety X in codimension two is compound Du Val, ii) it has a crepant resolution, iii) it is analytically a product of a rational double point and a nonsingular curve when $p \geqslant 3$ with two exceptions in $p = 3$, and iv) $R^1\pi_*\mathcal{O}_{\tilde{X}} = R^1\pi_*K_{\tilde{X}} = 0$ holds outside some finite points of X for any resolution of singularities $\pi : \tilde{X} \to X$.

1. Introduction

The notion of canonical singularities was introduced by Reid in [22] as a higher dimensional analogue of rational double points, and played a very important role in the classification theory of algebraic varieties defined over an algebraically closed field of characteristic 0. In positive characteristic, the classification of rational double points was completed by M. Artin in [3], but nothing seems to be known in higher dimensional cases.

Much inspired by recent progress in the theory of resolutions of singularities of three dimensional algebraic varieties in positive characteristic, especially by Cutkosky [7], Cossart and Piltant [4,5], we

* Corresponding author.
E-mail addresses: hirokado@math.info.hiroshima-cu.ac.jp (M. Hirokado), ito_hiroyuki@ma.noda.tus.ac.jp (H. Ito), natsuo@math.info.hiroshima-cu.ac.jp (N. Saito).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.10.007

attempt to study three dimensional canonical singularities over an algebraically closed field of positive characteristic.

The following fundamental result was achieved by Elkik [10], Reid [22] and Shepherd–Barron (cf. [23]) over the field of complex numbers.

Theorem 1. *Let V be a normal algebraic variety of dimension three defined over the field of complex numbers with at most canonical singularities. Then the following assertions hold.*

i) *V has at most rational singularities,*
ii) *V is Cohen–Macaulay,*
iii) *V is either nonsingular or locally isomorphic to the ambient space of a trivial one-parameter deformation of a rational double point in codimension two.*

We try to generalize this result to threefolds defined over an algebraically closed field k of characteristic $p > 0$. When considering this problem, one finds that there lie three major obstacles as below:

a) The proof that a general hyperplane section of V has at most rational double points breaks down in positive characteristic, because of the lack of Bertini's theorem.
b) Mimicking the characteristic-zero-proof does not give that V is a Cohen–Macaulay scheme in our case, because of the lack of the Grauert–Riemenschneider vanishing theorem. Besides, it is still very difficult to understand the properties of canonical or index 1 covers of the singularity, when the mapping degree is divisible by p.
c) A flat family whose general fiber is a germ of a rational double point may no longer be locally a trivial deformation, "A rational double point may have moduli in positive characteristic".

As for the obstacle a), the next example shows that a general member can be nonnormal when we consider a base point free complete linear system of nef and big divisors of a nonsingular threefold.

Example 1. Consider a line ℓ in a projective space $\mathbf{P}^3$ defined over an algebraically closed field of characteristic $p > 0$. Then blow up $\pi_1 : X_1 \to \mathbf{P}^3$ along ℓ with the exceptional divisor E_1. Choose a nonsingular p^e-multisection $\sigma \subset E_1$ such that the morphism of $\pi_1|_\sigma : \sigma \to \ell$ is purely inseparable of degree p^e, and blow up $\pi_2 : X_2 \to X_1$ along σ, with the exceptional divisor E_2. Then blow up $\pi_3 : X_3 \to X_2$ along $E_1 \cap E_2$, with the exceptional divisor E_3. Then blow up $\pi_4 : X_4 \to X_3$ along $E_3 \cap E_2$, with the exceptional divisor E_4 and denote the whole morphism by $\pi : X_4 \to \mathbf{P}^3$. Then a general member $D \in |\pi^* \mathcal{O}_{\mathbf{P}^3}(1)|$ satisfies $D \cap E_4 \subset \mathrm{Sing}(D)$.

The following theorem gives some examples concerning the obstacle c). For the classification of rational double points in positive characteristic, see Artin [3].

Theorem 2. *(See Hirokado, Ito and Saito [13,14].) Let $X := \operatorname{Spec} k[[x, y, z, w]]/(f)$ be one of the hypersurface singularities in the table*:

Characteristic	*Equation f*	*Type*
$p = 3$	$x^3 + y^2 + z^3 w$	E_6^0
$p = 2$	$x^3 + y^2 + z^2 w$	D_4^0
	$x^2 + xy^2 + z^2 + z^2 w$	D_4^0
	$x^3 + y^2 + z^4 w$	E_8^0
	$x^2 + xy^2 + z^2 + z^4 w$	D_8^0
	$x^2 + xy^2 + z^2 + z^3 + z^3 w$	D_6^2
	$x^3 z + y^2 + z^2 w$	D_6^1

Then the following assertions hold.