Journal of Algebra 373 (2013) 223–248

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Effectively categorical abelian groups ☆

Rodney Downey [a], Alexander G. Melnikov [b,*]

[a] Department of Mathematics, Statistics, and Operations Research, Victoria University of Wellington, New Zealand
[b] School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore

ARTICLE INFO

Article history:
Received 30 June 2011
Available online 27 October 2012
Communicated by Efim Zelmanov

Keywords:
Abelian groups and modules
Computable model theory
Effective categoricity

ABSTRACT

We study effective categoricity of computable abelian groups of the form $\bigoplus_{i\in\omega} H$, where H is a subgroup of $(Q,+)$. Such groups are called homogeneous completely decomposable. It is well-known that a homogeneous completely decomposable group is computably categorical if and only if its rank is finite.

We study Δ^0_n-categoricity in this class of groups, for $n > 1$. We introduce a new algebraic concept of S-independence which is a generalization of the well-known notion of p-independence. We develop the theory of S-independent sets. We apply these techniques to show that every homogeneous completely decomposable group is Δ^0_3-categorical.

We prove that a homogeneous completely decomposable group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to the free module over the localization of Z by a computably enumerable set of primes P with the semi-low complement (within the set of all primes).

We apply these results and techniques to study the complexity of generating bases of computable free modules over localizations of integers, including the free abelian group.

☆ We are grateful to Andre Nies, Noam Greenberg, and the anonymous referee who suggested numerous corrections to the paper. Many thanks to Iskander Kalimullin for useful and motivative discussions at different stages of the project. We acknowledge support of the Marsden Fund for this project.

* Corresponding author.
E-mail addresses: rod.downey@msor.vuw.ac.nz (R. Downey), amelnikov@ntu.edu.sg (A.G. Melnikov).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.09.020

1. Introduction

1.1. Computable structures and effective categoricity

Remarkably, the study of effective procedures in group theory pre-dates the clarification of what is meant by a computable process; beginning at least with the work of Max Dehn in 1911 [8] who studied word, conjugacy and isomorphisms in finitely presented groups. While the original questions concerned themselves with finitely presented groups, it turned out that they were intrinsically connected with questions about infinite presentations with computable properties. In [22], Graham Higman proved what is now called the Higman Embedding Theorem which stated that a finitely generated group could be embedded into a finitely presented one iff it had a computable presentation (in a certain sense).

The current paper is centered in the line of research of effective procedures in computably presented groups. By computable groups, we mean groups where the domain is computable and the algebraic operation is computable upon that domain.

Such studies can be generalized to other algebraic structures such as fields, rings, vector spaces and the like, a tradition going back to Grete Hermann [21], van der Waerden [44], and explicitly using computability theory, Rabin [40], Mal'tsev [32] and Fröhlich and Shepherdson [17].

More generally, computably presentable algebraic structures are the main objects of study in computable model theory and effective algebra. Recall that for an infinite countable algebraic structure $\mathcal{A}$, a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is called a *computable presentation of* $\mathcal{A}$ if the domain of $\mathcal{B}$ is (coded by) $\mathbb{N}$, and the atomic diagram of $\mathcal{B}$ is a computable set. If a structure has a computable presentation then it is *computably presentable*. In the same way that *isomorphism* is the canonical classification tool in classical algebra, when we take presentations into account, *computable* isomorphism becomes the main tool. Now two presentations are regarded as the same if they agree up to computable isomorphism. However, an infinite computably presentable structure $\mathcal{A}$ may have many of different computable presentations. Such differing presentations reflect differing computational properties. For example, a computable copy of the order type of the natural numbers might have the successor relation computable (as the familiar presentation does), whereas another might have this successor relation non-computable. Such copies cannot be computably isomorphic.

An infinite countable structure $\mathcal{A}$ is *computably categorical* or *autostable* if every two computable presentations of $\mathcal{A}$ have a computable isomorphism between them. This would mean that the computability-theoretical properties of every copy are identical. Cantor's back-and-forth argument shows that the dense linear ordering without endpoints forms a computably categorical structure. Computable categoricity is one of the central notions of computable model theory (see [15] or [3]). For certain familiar classes of structures we can characterize computable categoricity by algebraic invariants. For instance, a computably presentable Boolean algebra is computably categorical exactly if it has only finitely many atoms [19,29], a computably presentable linear order is computably categorical if and only if it has only finitely many successive pairs [41], and a computably presentable torsion-free abelian group is computably categorical if and only if its rank is finite [20,39].

Computably categorical structures tend to be quite rare, and it is natural to ask the question of how close to being computably categorical a structure is. As mentioned above, we know that a linear ordering of order type $\mathbb{N}$ is not computably categorical since there is the canonical example where the successor relation is computable, and another where the successor relation is not. But if we are given an oracle for the successor relation, then the structure is computably categorical *relative* to that. The halting problem would be enough to decide whether y is the successor of x in such an ordering. This motivates the following definition.

We say that a structure $\mathcal{A}$ is Δ^0_n*-categorical* if every two computable presentations of $\mathcal{A}$ have an isomorphism between them which is computable with oracle $\emptyset^{(n-1)}$, where $\emptyset^{(n-1)}$ is the $(n-1)$-th iteration of the halting problem. Once computably categorical structures in a given class are characterized, it is natural to ask which members of this class are Δ^0_2-categorical. Here the situation becomes more complex. There are only few results in this area, most of them are partial. For instance, McCoy [34] characterizes Δ^0_2-categorical linear orders and Boolean algebras under some extra effectiveness conditions. Also it is known that in general Δ^0_{n+1}-categoricity does not imply Δ^0_n-categoricity