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New irreducible modules for Heisenberg and affine Lie algebras

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ABSTRACT

We study $\mathbb{Z}$-graded modules of nonzero level with arbitrary weight multiplicities over Heisenberg Lie algebras and the associated generalized loop modules over affine Kac–Moody Lie algebras. We construct new families of such irreducible modules over Heisenberg Lie algebras. Our main result establishes the irreducibility of the corresponding generalized loop modules providing an explicit construction of many new examples of irreducible modules for affine Lie algebras. In particular, to any function $\varphi : \mathbb{N} \to \{\pm\}$ we associate a φ-highest weight module over the Heisenberg Lie algebra and a φ-imaginary Verma module over the affine Lie algebra. We show that any φ-imaginary Verma module of nonzero level is irreducible.

1. Introduction

Affine Lie algebras are the most studied among the infinite-dimensional Kac–Moody Lie algebras and have widespread applications. Their representation theory is far richer than that of finite-dimensional simple Lie algebras. In particular, affine Lie algebras have irreducible modules containing both finite- and infinite-dimensional weight spaces, something that cannot happen in the finite-dimensional setting. These representations arise from taking non-standard partitions of the root system; that is, partitions which are not equivalent under the Weyl group to the standard partition into positive and negative roots (see [DFG]). For affine Lie algebras, there are always only finitely many equivalence classes of such non-standard partitions (see [F4]). Corresponding to each partition is a Borel subalgebra, and one can form representations induced from one-dimensional modules for

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these Borel subalgebras. These modules, often referred to as *Verma-type modules*, were first studied by Jakobsen and Kac [JK], and by Futorny [F3,F4]. Results on the structure of Verma-type modules can also be found in [Co,F1,FS].

Let $\mathfrak{g}$ be an affine Lie algebra, $\mathfrak{h}$ its standard Cartan subalgebra, and $\mathfrak{z} = \mathbb{C}c$ its center, where c is the canonical central element. Let V be a weight $\mathfrak{g}$-module, that is, $\mathsf{V} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathsf{V}_\mu$, where $\mathsf{V}_\mu = \{v \in \mathsf{V} \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. If V is irreducible, then c acts as a scalar on V called the *level* of V. The theory of Verma-type modules is best developed in the case when the level is nonzero [F4]. For example, the so-called imaginary Verma modules induced from the natural Borel subalgebra are always irreducible when the level is nonzero [JK,F2].

The classification of irreducible modules is known only for modules with finite-dimensional weight spaces (see [FT]) and for certain subcategories of induced modules with some infinite-dimensional weight spaces (see for example, [F3,FKM,FK]). Our main goal is to go beyond the modules with finite-dimensional weight spaces and to construct new irreducible modules of nonzero level with infinite-dimensional weight spaces. Examples of such modules have been constructed previously by Chari and Pressley in [CP] as the tensor product of highest and lowest weight modules.

Here we consider different Borel-type subalgebras that do not correspond to partitions of the root system of $\mathfrak{g}$. Such a subalgebra is determined by a function $\varphi : \mathbb{N} \to \{\pm\}$ on the set $\mathbb{N}$ of positive integers, and so is denoted $\mathfrak{b}_\varphi$. The subalgebra $\mathfrak{b}_\varphi$ gives rise to a class of $\mathfrak{g}$-modules called *φ-imaginary Verma modules*. These modules can be viewed as induced from φ-highest weight modules over the Heisenberg subalgebra of $\mathfrak{g}$. This construction is similar to the construction of imaginary Whittaker modules in [Ch], but unlike the modules in [Ch], our modules over the Heisenberg subalgebra are $\mathbb{Z}$-graded. If $\varphi(n) = +$ for all $n \in \mathbb{N}$, then $\mathfrak{b}_\varphi$ is the natural Borel subalgebra of $\mathfrak{g}$.

We establish a criterion for the irreducibility of φ-imaginary Verma modules. It comes as no surprise that any such module is irreducible if and only if it has a nonzero level.

Next we consider the classification problem for irreducible $\mathbb{Z}$-graded modules for the Heisenberg subalgebra of $\mathfrak{g}$. The ones of level zero were determined by Chari [C]. Any such module of nonzero level with a $\mathbb{Z}$-grading has all its graded components infinite-dimensional by [F1]; otherwise, it is a highest weight module. We classify all *admissible diagonal* $\mathbb{Z}$-graded irreducible modules of nonzero level for an arbitrary infinite-dimensional Heisenberg Lie algebra. Since the $\mathbb{Z}$-graded components of a module are not assumed to be finite-dimensional the restriction on a module to be diagonal is natural. We show that these modules have a $\mathbb{Z}^\infty$-gradation and can be obtained from weight modules over an associated Weyl algebra as in [BBF] by compression of the gradation.

The restriction on a module to be admissible leads to an equivalence between the category of admissible diagonal $\mathbb{Z}$-graded modules for Heisenberg Lie algebras and the category of admissible weight modules for the Weyl algebra A_∞ [BBF]. Examples of such modules were considered earlier by Casati [Ca], where they were constructed by means of an action of differential operators on a space of polynomials in infinitely many variables (compare Theorem 4.21 below). Examples of non-admissible diagonal $\mathbb{Z}$-graded irreducible modules were constructed in [MZ].

We use parabolic induction to construct generalized loop modules for the affine Lie algebra $\mathfrak{g}$. The modules are induced from an arbitrary irreducible $\mathbb{Z}$-graded module of nonzero level for the Heisenberg subalgebra of $\mathfrak{g}$. This construction extends Chari's construction to the nonzero level case. Our main result establishes the irreducibility of any generalized loop module induced from a diagonal irreducible module of nonzero level for the Heisenberg subalgebra. By this process, we obtain new families of irreducible modules of nonzero level for any affine Lie algebra. The irreducible modules constructed in [Ca] are "dense" in the sense that they have the maximal possible set of weights and hence are different from the ones studied here.

It should be noted that all results in our paper hold for both the untwisted and twisted affine Lie algebras.

The structure of the paper is as follows. In Section 3, we construct φ-imaginary Verma modules for affine Lie algebras for any function $\varphi : \mathbb{N} \to \{\pm\}$, and establish a criterion for their irreducibility in Theorem 3.5. In Section 4, we consider various types of modules (torsion, locally-finite, diagonal, and admissible) for the Heisenberg algebra. Theorems 4.5 and 4.15 (see Corollary 4.16) provide the classification of all irreducible $\mathbb{Z}$-graded admissible diagonal modules of nonzero level for the Heisenberg algebra. Finally, in Section 5, we introduce generalized loop modules for affine Lie algebras and study their structure. These modules are induced from finitely generated $\mathbb{Z}$-graded irreducible