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Simplicity and maximal commutative subalgebras of twisted generalized Weyl algebras

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ABSTRACT

In this paper we prove three theorems about twisted generalized Weyl algebras (TGWAs). First, we show that each non-zero ideal of a TGWA has non-zero intersection with the centralizer of the distinguished subalgebra R. This is analogous to earlier results known to hold for crystalline graded rings. Second, we give necessary and sufficient conditions for the centralizer of R to be commutative (hence maximal commutative), generalizing a result by V. Mazorchuk and L. Turowska. And third, we generalize results by D.A. Jordan and V. Bavula on generalized Weyl algebras by giving necessary and sufficient conditions for certain TGWAs to be simple, in the case when R is commutative. We illustrate our theorems by considering some special classes of TGWAs and providing concrete examples. We also discuss how simplicity of a TGWA is related to the maximal commutativity of R and the (non-)existence of non-trivial $\mathbb{Z}^n$-invariant ideals of R.

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1. Introduction

Higher rank generalized Weyl algebras (GWAs) were introduced by Bavula in [1] and can be defined as follows. Let R be a ring (from Section 3 and onwards, R is always assumed to be *commutative*) and $\sigma : \mathbb{Z}^n \to \mathrm{Aut}(R)$, $g \mapsto \sigma_g$, an action of $\mathbb{Z}^n$ on R by ring automorphisms, $t = (t_1, \ldots, t_n)$ an n-tuple of non-zero elements of the center of R such that $\sigma_i(t_j) = t_j$ for all $i \neq j$. The *generalized Weyl algebra of rank* (or *degree*) $n \in \mathbb{Z}_{>0}$, denoted $R(\sigma, t)$, is the ring extension of R by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ subject to the relations

$$X_i r = \sigma_i(r) X_i, \qquad Y_i r = \sigma_i^{-1}(r) Y_i, \quad \forall r \in R,\ i \in \{1, \ldots, n\}, \tag{1a}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i), \quad \forall i \in \{1, \ldots, n\}, \tag{1b}$$

$$Y_i Y_j = Y_j Y_i, \qquad X_i X_j = X_j X_i, \quad \forall i, j \in \{1, \ldots, n\}, \tag{1c}$$

$$X_i Y_j = Y_j X_i, \quad \forall i \neq j, \tag{1d}$$

where $\sigma_i := \sigma_{e_i}$ and $e_i = (0, \ldots, \overset{i}{1}, \ldots, 0) \in \mathbb{Z}^n$. In [16] Mazorchuk and Turowska introduced a class of algebras called *twisted generalized Weyl algebras* (TGWAs), which is a generalization of the higher rank generalized Weyl algebras, in the case when none of the t_i's are zero-divisors in R (see Section 2.3 for the precise definition). Basically relation (1c) is relaxed in order to also accommodate situations where the defining relations among the X_i's (and among the Y_i's) are some q-commutation relations (say, $X_i X_j = q X_j X_i$, $i < j$) or some Serre-type relations (such as $X_i^2 X_j - 2X_i X_j X_i + X_j X_i^2 = 0$). Note that when some of the t_i's are zero-divisors, only certain proper quotients of the GWA can be obtained as a TGWA (cf. Theorem 4.3, whose assertion (iii) always holds for a GWA).

Already GWAs of rank one include many interesting algebras such as $U(\mathfrak{sl}(2))$ and its quantizations and deformations, as well as several quantized function spaces and so-called generalized down–up algebras, see for example [3] and references therein. This unification of different algebras into one family has proved to be very fruitful. For example in [6] the authors classified all simple and indecomposable weight modules over a rank one GWA, which in particular gives in one stroke a description of such modules for characteristic zero, quantized, and modular $U(\mathfrak{sl}(2))$.

In higher rank several algebras which one would expect to be GWAs, such as the multiparameter quantized Weyl algebras defined in [13], are not (at least not in an obvious way). However, they are TGWAs as was shown in [17]. In [15] some algebras of importance in the representation theory of $\mathfrak{gl}(n)$, namely the Mickelsson–Zhelobenko algebra $Z(\mathfrak{gl}(n), \mathfrak{gl}(n-1))$ and the extended orthogonal Gelfand–Tsetlin algebras (including certain localizations of $U(\mathfrak{gl}(n))$), were shown to be examples of TGWAs. It is expected that the corresponding quantized versions of these algebras are also TGWAs. Sergeev [26] proved that some primitive quotients of $U(\mathfrak{gl}(3))$ are TGWAs and used this fact to produce multivariable analogs of Hahn polynomials.

When it comes to representation theory, several classes of simple weight (with respect to the subring R, assumed to be commutative) modules over TGWAs have been classified in [9,15,16], including simple graded modules [15]. Bounded and unbounded $*$-representations of TGWAs were described in [17].

In this paper we investigate TGWAs from another point of view, namely that of graded algebras; every TGWA of rank n is $\mathbb{Z}^n$-graded in a natural way. It is known that the class of higher rank GWAs, hence the class of TGWAs, includes all skew group algebras over $\mathbb{Z}^n$ (take $t_i = 1$ for all i in the definition above). Another related fact, proved in [8], is that any TGWA where t_i is regular for each i can be embedded in a crossed product algebra.

Consider the following problem.

Problem 1.1. Let G be a group with identity element denoted e. Given a G-graded algebra $A = \bigoplus_{g \in G} A_g$ such that A_e is commutative, is it true that each non-zero ideal of A has non-zero intersection with the centralizer of A_e in A?