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Closed subsets in duals of commutative table algebras

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ABSTRACT

A lower bound on the number of elements outside a closed subset of a C-basis of a commutative C-algebra with dual a table algebra is derived, as is an equivalent condition for when the lower bound is met. As corollaries, lower bounds are obtained on the number of primitive idempotent matrices of rank greater than 1 in the adjacency algebra of a commutative, imprimitive association scheme; and, for a given normal subgroup N of a finite group G, on the number of irreducible characters of G whose kernels do not contain N, and on the number of conjugacy classes of G not contained in N. Also found are equivalent conditions for when these lower bounds are met.

1. Introduction

The aim of this paper is to produce and characterize a lower bound on the number of elements outside a closed subset of a C-basis of a commutative C-algebra whose dual is a table algebra. For a finite group, the center of the group algebra over the complex numbers, with basis the conjugacy class sums, and the algebra of complex-valued class functions, with basis the irreducible characters, are commutative table algebras that are dual to each other as C-algebras, in the sense of [7, Definition 1.5]. Because of this, we easily obtain Theorems A, C, D in [4] as corollaries of our main results. The adjacency algebra of a commutative association scheme is another C-algebra whose dual is a table algebra [5, Sections II.2, II.3]. Thus we obtain, as another consequence, an evidently new theorem on the primitive idempotent matrices in such adjacency algebras (Theorem 1.8 below).

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We first provide some notation, and then state an elementary result about matrices. The theorem yields, via the analysis of eigenmatrices, a direct route to the main results and subsequent applications. This approach is quite different from that of [4].

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^{\times}$ and $\mathbb{R}^{+}$ denote respectively the set of complex numbers, the set of real numbers, the set of nonzero real numbers, and the set of positive real numbers. For any $\alpha \in \mathbb{C}$, α^* denotes the complex conjugate of α. For a set S, denote the cardinality of S by $|S|$.

Let M be an $n \times m$ matrix over $\mathbb{C}$. Fix positive integers n_1, n_2, m_1, m_2 with $n_1 + n_2 = n$ and $m_1 + m_2 = m$. Partition M into submatrices M_{tl} (top left: $n_1 \times m_1$), M_{tr} (top right: $n_1 \times m_2$), M_{bl} (bottom left: $n_2 \times m_1$), M_{br} (bottom right: $n_2 \times m_2$), as shown:

$$M = \begin{array}{cc} & \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \left(\begin{array}{c|c} M_{tl} & M_{tr} \\ \hline M_{bl} & M_{br} \end{array} \right) \end{array}.$$

Let M_t $(n_1 \times m) := (M_{tl}|M_{tr})$ and M_b $(n_2 \times m) := (M_{bl}|M_{br})$.

Theorem 1.1. *Let M be an $n \times n$ matrix over $\mathbb{C}$, partitioned as M_{tl}, M_{tr}, M_{bl}, M_{br} as above, where n_1, n_2, m_1, m_2 are positive integers with $n_1 + n_2 = n = m_1 + m_2$. Fix $\varepsilon_j \in \mathbb{R}^{+}$ for $1 \leqslant j \leqslant n$, and let $\langle\ ,\ \rangle$ be the positive definite Hermitian form on $\mathbb{C}^n$ defined as follows: for $x = (\xi_j)_{j=1}^n$, $y = (\eta_j)_{j=1}^n \in \mathbb{C}^n$ (so $\xi_j, \eta_j \in \mathbb{C}$), and where η_j^* denotes the complex conjugate of η_j,*

$$\langle x, y \rangle = \sum_{j=1}^{n} \xi_j \eta_j^* \varepsilon_j.$$

Assume that the columns of M are nonzero, and are mutually orthogonal with respect to $\langle\ ,\ \rangle$. Let p denote the number of distinct columns of M_{tl}. Then

(i) $n_2 = \operatorname{rank}(M_b) \geqslant m_1 - p$;
(ii) *if $n_2 = m_1 - p$ then $M_{br} = 0$;*
(iii) *if $M_{br} = 0$ and the distinct columns of M_{tl} are linearly independent, then $n_2 = m_1 - p$.*

The proof of Theorem 1.1 is given in Section 2 below.

Definition 1.1. (See [7, Definition 1.1].) Let $\mathbf{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis of a finite dimensional associative commutative algebra A over $\mathbb{C}$, with identity element $1_A = b_1$. Let $f : A \to \mathbb{C}$ be an algebra homomorphism. Then the triple $(A, \mathbf{B}, f)$ is a commutative *C-algebra* (and $\mathbf{B}$ is a *C-basis*) if

(i) For all i, j, m, $b_i b_j = \sum_{m=1}^{m=n} \beta_{ijm} b_m$ with $\beta_{ijm} \in \mathbb{R}$.
(ii) There is an algebra automorphism (denoted by $^-$) of A whose order divides 2, such that $b_i \in \mathbf{B}$ implies $\bar{b}_i \in \mathbf{B}$ (then $\bar{i}$ is defined by $b_{\bar{i}} = \bar{b}_i$).
(iii) For all i, j, $\beta_{ij1} \neq 0$ if and only if $j = \bar{i}$; and $\beta_{i\bar{i}1} > 0$.
(iv) For all i, $f(b_i) = f(\bar{b}_i) \in \mathbb{R}^{\times}$.

Definition 1.2. The β_{ijm}, for $i, j, m = 1, \ldots, n$, are *the structure constants* of $\mathbf{B}$. We call the nonzero real numbers $f(b_i)$ the *degrees* of $(A, \mathbf{B}, f)$.

Definition 1.3. (See [7, Definition 1.2].) Let $\mathbf{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis of a finite dimensional associative commutative algebra A over $\mathbb{C}$, with identity element $1_A = b_1$. If Definition 1.1(i)–(iii) all hold, with $\beta_{ijm} \in \mathbb{R}^{+} \cup \{0\}$ for all i, j, m, then $(A, \mathbf{B})$ is called a commutative *table algebra*.