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The proportion of Weierstrass semigroups

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ABSTRACT

We solve a problem of Komeda concerning the proportion of numerical semigroups which do not satisfy Buchweitz' necessary criterion for a semigroup to occur as the Weierstrass semigroup of a point on an algebraic curve. A result of Eisenbud and Harris gives a sufficient condition for a semigroup to occur as a Weierstrass semigroup. We show that the family of semigroups satisfying this condition has density zero in the set of all semigroups. In the process, we prove several more general results about the structure of a typical numerical semigroup.

1. Introduction

A numerical semigroup S is an additive submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. The complement is referred to as the gap set and is denoted by $H(S)$. Its size is called the genus of S and is usually denoted by $g(S)$. The largest of these gaps is called the Frobenius number, denoted $F(S)$, and the smallest nonzero nongap is called the multiplicity, denoted $m(S)$. When it will not cause confusion we will omit the S and write g, F and m. A very good source for background on numerical semigroups is [9].

Let C be a smooth projective algebraic curve of genus g over the complex numbers. It is a theorem of Weierstrass that given any $p \in C$ there are exactly g integers $\alpha_i(p)$ with $1 = \alpha_1(p) < \cdots < \alpha_g(p) \leqslant 2g - 1$ such that there does not exist a meromorphic function f on C which has a pole of order $\alpha_i(p)$ at p and no other singularities [5]. This characterization makes it clear that the set $\mathbb{N} \setminus \{\alpha_1(p), \dots, \alpha_g(p)\}$ is a numerical semigroup of genus g. We say that a semigroup S is Weierstrass

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if there exist some curve C and some point $p \in C$ such that S is this semigroup. In the late 19th century, Hurwitz suggested studying which numerical semigroups are Weierstrass [8].

A point p such that $(\alpha_1(p), \ldots, \alpha_g(p)) \neq (1, \ldots, g)$ is called a Weierstrass point of C, and it is known that there are at most $g^3 - g$ such points. It is an active area of research to consider a multiset $\mathcal{S}$ of at most $g^3 - g$ semigroups of genus g and study the set of curves for which $\mathcal{S}$ is the collection of semigroups of the Weierstrass points of the curve. This multiset gives us important information about the geometry of the curve. We would like to better understand, for example, the dimension of the moduli space of curves with a fixed collection of semigroups attached to their Weierstrass points. For more on the history of this problem see the article of del Centina [5], or the book [1].

In this paper we focus on two criteria that address this problem of Hurwitz. The first is a simple combinatorial criterion of Buchweitz which is necessary for a semigroup to occur as the Weierstrass semigroup of some point on some curve C [4]. This condition gave the first proof that not all semigroups are Weierstrass. In the final section of the paper we consider a criterion of Eisenbud and Harris [6], which shows that certain semigroups are Weierstrass. These two simple criteria cover much of what we know about this problem. The main result of this paper is to show that in some sense, both of the sets covered by these criteria have density zero in the entire set of numerical semigroups. The overall proportion of Weierstrass semigroups remains completely unknown.

Let $N(g)$ be the number of numerical semigroups of genus g. Recent work of Zhai [15], building on work of Zhao [16], gives a better understanding of the growth of $N(g)$. These papers build towards resolving a conjecture of Bras-Amorós [3].

Theorem 1 *(Zhai). The function $N(g)$ satisfies*

$$\lim_{g\to\infty} \frac{N(g)}{\varphi^g} = C,$$

where $C > 0$ is a constant and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

This result will play an important role in some of our proofs.

We next recall the criterion of Buchweitz [4].

Proposition 2 *(Buchweitz). Let S be a semigroup of genus g and let $H(S)$ be the set of gaps of S. Suppose that there exists some $n > 1$ such that*

$$\left|nH(S)\right| > (2n-1)(g-1),$$

where $nH(S)$ is the n-fold sum of the set $H(S)$. Then S is not Weierstrass.

Let $NB(g)$ be the number of semigroups S of genus g for which there is some n such that S does not satisfy the Buchweitz criterion with this n. Let $NB_2(g)$ be the number of semigroups S of genus g such that $|2H(S)| > 3(g-1)$. Komeda seems to be the first to have studied $\lim_{g\to\infty} \frac{NB_2(g)}{N(g)}$ [11].

The following is part of a table included in [11]:

g	$N(g)$	$NB_2(g)$	$\frac{NB_2(g)}{N(g)}$
16	4806	2	.000416
17	8045	6	.000746
18	13 467	15	.001114
19	22 464	31	.001380
20	37 396	67	.001792
21	62 194	145	.002331
22	103 246	293	.002838
23	170 963	542	.003170
24	282 828	1053	.003723
25	467 224	1944	.004161