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The primary components of positive critical binomial ideals

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ABSTRACT

A natural candidate for a generating set of the (necessarily prime) defining ideal of an n-dimensional monomial curve, when the ideal is an almost complete intersection, is a full set of n critical binomials. In a somewhat modified and more tractable context, we prove that, when the exponents are all positive, critical binomial ideals in our sense are not even unmixed for $n \geqslant 4$, whereas for $n \leqslant 3$ they are unmixed. We further give a complete description of their isolated primary components as the defining ideals of monomial curves with coefficients. This answers an open question on the number of primary components of Herzog–Northcott ideals, which comprise the case $n = 3$. Moreover, we find an explicit, concrete description of the irredundant embedded component (for $n \geqslant 4$) and characterize when the hull of the ideal, i.e., the intersection of its isolated primary components, is prime. Note that these last results are independent of the characteristic of the ground field. Our techniques involve the Eisenbud–Sturmfels theory of binomial ideals and Laurent polynomial rings, together with theory of Smith Normal Form and of Fitting ideals. This gives a more transparent and completely general approach, replacing the theory of multiplicities used previously to treat the particular case $n = 3$.

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1. Introduction

Let $A = k[x_1, \dots, x_n]$ be the polynomial ring in n variables $\underline{x} = x_1, \dots, x_n$ over a field k, $n \geqslant 2$. Set $\mathfrak{m} = (\underline{x})$, the maximal ideal generated by the x_i. Let $a_{i,j} \in \mathbb{N}$, $i, j = 1, \dots, n$, with $a_{i,i} = \sum_{j \neq i} a_{i,j}$, and let L be the $n \times n$ integer matrix defined as follows:

$$L = \begin{pmatrix} a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & a_{n,n} \end{pmatrix},$$

where the sum of the entries of each row is zero. (Here $\mathbb{N}$ denotes the set of positive integers.) For ease of reference, L will be called a *positive critical binomial matrix* (PCB matrix, for short). Set $d \in \mathbb{N}$ to be the greatest common divisor of the $(n-1) \times (n-1)$ minors of L. (We shall see below that these minors are non-zero.) Let $\underline{f} = f_1, \dots, f_n$ be the binomials defined by the columns of L:

$$f_1 = x_1^{a_{1,1}} - x_2^{a_{2,1}} \cdots x_n^{a_{n,1}}, \qquad f_2 = x_2^{a_{2,2}} - x_1^{a_{1,2}} x_3^{a_{3,2}} \cdots x_n^{a_{n,2}}, \qquad \dots, \qquad f_n = x_n^{a_{n,n}} - x_1^{a_{1,n}} \cdots x_{n-1}^{a_{n-1,n}}.$$

Let $I = (\underline{f})$ be the binomial ideal generated by the f_j. We will call I the *positive critical binomial ideal* (PCB ideal, for short) associated to L.

The purpose of this paper is to investigate the primary decomposition of PCB ideals and to contrast this theory with analogous results in [OP2] concerning ideals of Herzog–Northcott type, which comprise the case $n = 3$. We first prove that, if $n \geqslant 4$ (respectively, $n \leqslant 3$), I has at most $d + 1$ (respectively, d) primary components. This answers a question posed in [OP2, Remark 8.6].

We will observe that I is contained in a unique toric ideal $\mathfrak{p}_m$ associated to the monomial curve $\Gamma_m = \{(\lambda^{m_1}, \dots, \lambda^{m_n}) \in \mathbb{A}_k^n \mid \lambda \in k\}$, where $m = (m_1, \dots, m_n) = m(I) \in \mathbb{N}^n$ is determined by I. That is, $\mathfrak{p}_m$ (referred to as the *Herzog* ideal associated to m) is the kernel of the natural homomorphism $A \to k[t]$, t a variable over k, that sends each x_i to t^{m_i}.

In somewhat more detail, if k contains the d-th roots of unity and the characteristic of k, $\mathrm{char}(k)$, is zero or $\mathrm{char}(k) = p$, p a prime with $p \nmid d$, we give a full description of a minimal primary decomposition of I. Namely, the intersection of the isolated primary components of I, $\mathrm{Hull}(I)$, is equal to the intersection of d prime toric ideals of "monomial curves with coefficients", i.e., kernels of natural homomorphisms $A \to k[t]$ that send each x_i to $\lambda_i t^{m_i}$, $\lambda_i \in k$. This will explain the "intrinsic" role of the Herzog ideal $\mathfrak{p}_{m(I)}$ among the other minimal primes of I as the instance where each of the "coefficients" λ_i equals 1.

Furthermore, if $n \leqslant 3$, I is unmixed and $I = \mathrm{Hull}(I)$. But if $n \geqslant 4$, I has one irredundant embedded $\mathfrak{m}$-primary component. This provides a very striking contrast between the cases $n \leqslant 3$ and $n \geqslant 4$. In each case we give a concrete description of these primary components (cf. Theorems 4.10 and 7.1).

We now recall briefly from [OP2] some relevant parts of the theory of ideals of Herzog–Northcott type (or HN ideals, as they are referred to). The study of HN ideals had their origin in work of Herzog [Her] on the defining ideals $\mathfrak{p}_m$ of monomial space curves Γ_m, $m \in \mathbb{N}^3$, $\gcd(m) = 1$. The ideals $\mathfrak{p}_m$, which are Cohen–Macaulay almost complete intersection ideals of height two, proved useful in work of the authors in settling a long-standing open question on an aspect of the uniform Artin–Rees property (cf. [OP1]); this work built on the observation that these ideals $\mathfrak{p}_m$ were a particular case of a class of ideals studied by Northcott [Nor].

In [OP2] we defined an HN ideal I as the determinantal ideal generated by the 2×2 minors of a certain matrix. One can easily check that HN ideals and PCB ideals are two notions that coincide when $A = k[\underline{x}]$ with $n = 3$. In [OP2, Definition 7.1 and Remark 7.2] we introduced an integer vector $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ associated to I. We showed that I is prime if and only if the greatest common divisor of $m(I)$ is equal to 1 [OP2, Theorem 7.8]. Further, using techniques from the theory of multiplicities, we gave upper bounds for the number of prime components of I in terms of the m_i and $\gcd(m(I))$. Finally, using a Jacobian criterion, we showed that I is radical if the characteristic of k