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Integer-valued polynomials on algebras

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ABSTRACT

Let D be a domain with quotient field K and A a D-algebra. A polynomial with coefficients in K that maps every element of A to an element of A is called integer-valued on A. For commutative A we also consider integer-valued polynomials in several variables. For an arbitrary domain D and I an arbitrary ideal of D we show I-adic continuity of integer-valued polynomials on A. For Noetherian one-dimensional D, we determine spectrum and Krull dimension of the ring $\mathrm{Int}_D(A)$ of integer-valued polynomials on A. We do the same for the ring of polynomials with coefficients in $M_n(K)$, the K-algebra of $n \times n$ matrices, that map every matrix in $M_n(D)$ to a matrix in $M_n(D)$.

1. Introduction

Let D be a domain with quotient field K and A a D-algebra, such as, for instance, a group ring $D(G)$ or the matrix algebra $M_n(D)$.

We are interested in the rings of polynomials

$$\mathrm{Int}_D(A) = \{ f \in K[x] \mid f(A) \subseteq A \},$$

and, if A is commutative,

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$$\mathrm{Int}_D^n(A) = \{f \in K[x_1, \ldots, x_n] \mid f(A^n) \subseteq A\}.$$

Elements of the D-algebra A are plugged into polynomials with coefficients in K via the canonical homomorphism $\iota_A : A \to K \otimes_D A$, $\iota_A(a) = 1 \otimes a$.

In the special case $A = D$ these rings are known as rings of integer-valued polynomials, cf. [3]. They provide natural examples of non-Noetherian Prüfer rings [5,11], and have been used for proving results on the n-generator property in Prüfer rings [2]. Also, integer-valued polynomials are useful for polynomial interpolation of functions from D to D [8,4], and satisfy other interesting algebraic conditions such as analogues of Hilbert's Nullstellensatz [3,9].

These desirable properties of rings of integer-valued polynomials have motivated the generalization to polynomials with coefficients in K acting on a D-algebra A [10,12]. So far, not much is known about rings of integer-valued polynomials on algebras. We know that they behave somewhat like the classical rings of integer-valued polynomials if the D-algebra A is commutative. For instance, Loper and Werner [12] have shown that $\mathrm{Int}_{\mathbb{Z}}(\mathcal{O}_K)$ is Prüfer. If A is non-commutative, however, the situation is radically different. For instance, $\mathrm{Int}_{\mathbb{Z}}(M_2(\mathbb{Z}))$ is not Prüfer [12], and is far from allowing interpolation [10].

We will describe the spectrum of $\mathrm{Int}_D(A)$, for a one-dimensional Noetherian ring D and a finitely generated torsion-free D-algebra A, in the hope that this will facilitate further research. We will investigate more closely the special case of $A = \mathrm{M}_n(D)$: we determine a polynomially dense subset of $\mathrm{M}_n(D)$ and describe the image of a given matrix under the ring $\mathrm{Int}_D(\mathrm{M}_n(D))$.

A different ring of integer-valued polynomials on the matrix algebra $M_n(D)$, consisting of polynomials with coefficients in $M_n(K)$ that map matrices in $M_n(D)$ to matrices in $M_n(D)$, has been introduced by Werner [13]. We will show that it is isomorphic to the algebra of $n \times n$ matrices over "our" ring $\mathrm{Int}_D(\mathrm{M}_n(D))$ of integer-valued polynomials on $M_n(D)$ with coefficients in K.

Before we give a precise definition of the kind of D-algebra A for which we will investigate $\mathrm{Int}_D(A)$, a few examples. D is always a domain with quotient field K, and not a field.

1.1. Example. For fixed $n \in \mathbb{N}$, let $A = \mathrm{M}_n(D)$ be the D-algebra of $n \times n$ matrices with entries in D and

$$\mathrm{Int}_D(\mathrm{M}_n(D)) = \{f \in K[x] \mid \forall C \in \mathrm{M}_n(D)\colon\ f(C) \in \mathrm{M}_n(D)\}.$$

1.2. Example. Let $\mathrm{H} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ be the $\mathbb{Q}$-algebra of rational quaternions, $\mathrm{L} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ the $\mathbb{Z}$-subalgebra of Lipschitz quaternions, and

$$\mathrm{Int}_{\mathbb{Z}}(\mathrm{L}) = \{f \in \mathbb{Q}[x] \mid \forall z \in \mathrm{L}\colon\ f(z) \in \mathrm{L}\}.$$

1.3. Example. Let G be a finite group, $K(G)$ and $D(G)$ the respective group rings, and

$$\mathrm{Int}_D(D(G)) = \{f \in K[x] \mid \forall z \in D(G)\colon\ f(z) \in D(G)\}.$$

If G is commutative, we also consider

$$\mathrm{Int}_D^n(D(G)) = \{f \in K[x_1, \ldots, x_n] \mid \forall z \in D(G)^n\colon\ f(z) \in D(G)\},$$

for $n \in \mathbb{N}$, where $D(G)^n = D(G) \times \cdots \times D(G)$ denotes the Cartesian product of n copies of $D(G)$.

1.4. Example. Let $D \subseteq A$ be Dedekind rings with quotient fields $K \subseteq F$, and

$$\mathrm{Int}_D^n(A) = \{f \in K[x_1, \ldots, x_n] \mid f(A^n) \subseteq A\}.$$