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The classification of normalizing groups

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ABSTRACT

Let X be a finite set such that $|X| = n$. Let $\mathcal{T}_n$ and $\mathcal{S}_n$ denote the transformation monoid and the symmetric group on n points, respectively. Given $a \in \mathcal{T}_n \setminus \mathcal{S}_n$, we say that a group $G \leqslant \mathcal{S}_n$ is *a-normalizing* if

$$\langle a, G\rangle \setminus G = \langle g^{-1}ag \mid g \in G\rangle,$$

where $\langle a, G\rangle$ and $\langle g^{-1}ag \mid g \in G\rangle$ denote the subsemigroups of $\mathcal{T}_n$ generated by the sets $\{a\} \cup G$ and $\{g^{-1}ag \mid g \in G\}$, respectively. If G is a-normalizing for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$, then we say that G is *normalizing*.

The goal of this paper is to classify the normalizing groups and hence answer a question of Levi, McAlister, and McFadden. The paper ends with a number of problems for experts in groups, semigroups and matrix theory.

1. Introduction and preliminaries

For notation and basic results on group theory we refer the reader to [8,11]; for semigroup theory we refer the reader to [17]. Let $\mathcal{T}_n$ and $\mathcal{S}_n$ denote the monoid consisting of mappings from $[n] := \{1, \ldots, n\}$ to $[n]$ and the symmetric group on $[n]$ points, respectively. The monoid $\mathcal{T}_n$ is usually called the full transformation semigroup. In [21], Levi and McFadden proved the following result.

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Theorem 1.1. *Let $a \in \mathcal{T}_n \setminus \mathcal{S}_n$. Then*

(1) $\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle$ *is idempotent generated*;
(2) $\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle$ *is regular.*

Using a beautiful argument, McAlister [26] proved that the semigroups $\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle$ and $\langle a, \mathcal{S}_n \rangle \setminus \mathcal{S}_n$ (for $a \in \mathcal{T}_n \setminus \mathcal{S}_n$) have exactly the same set of idempotents; therefore, as $\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle$ is idempotent generated, it follows that

$$\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle = \langle a, \mathcal{S}_n \rangle \setminus \mathcal{S}_n.$$

Later, Levi [22] proved that $\langle g^{-1}ag \mid g \in \mathcal{S}_n \rangle = \langle g^{-1}ag \mid g \in \mathcal{A}_n \rangle$ (for $a \in \mathcal{T}_n \setminus \mathcal{S}_n$), and hence the three results above remain true when we replace $\mathcal{S}_n$ by $\mathcal{A}_n$. The following list of problems naturally arises from these considerations.

(1) Classify the groups $G \leqslant \mathcal{S}_n$ such that for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ we have that the semigroup $\langle g^{-1}ag \mid g \in G \rangle$ is idempotent generated.
(2) Classify the groups $G \leqslant \mathcal{S}_n$ such that for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ we have that the semigroup $\langle g^{-1}ag \mid g \in G \rangle$ is regular.
(3) Classify the groups $G \leqslant \mathcal{S}_n$ such that for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ we have

$$\langle a, G \rangle \setminus G = \langle g^{-1}ag \mid g \in G \rangle.$$

The two first questions were solved in [4] as follows:

Theorem 1.2. *If $n \geqslant 1$ and G is a subgroup of $\mathcal{S}_n$, then the following are equivalent*:

(i) *The semigroup $\langle g^{-1}ag \mid g \in G \rangle$ is idempotent generated for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$.*
(ii) *One of the following is valid for G and n*:
  (a) $n = 5$ *and G is* $\mathrm{AGL}(1,5)$;
  (b) $n = 6$ *and G is* $\mathrm{PSL}(2,5)$ *or* $\mathrm{PGL}(2,5)$;
  (c) *G is $\mathcal{A}_n$ or $\mathcal{S}_n$.*

Theorem 1.3. *If $n \geqslant 1$ and G is a subgroup of $\mathcal{S}_n$, then the following are equivalent*:

(i) *The semigroup $\langle g^{-1}ag \mid g \in G \rangle$ is regular for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$.*
(ii) *One of the following is valid for G and n*:
  (a) $n = 5$ *and G is C_5, D_5, or* $\mathrm{AGL}(1,5)$;
  (b) $n = 6$ *and G is* $\mathrm{PSL}(2,5)$ *or* $\mathrm{PGL}(2,5)$;
  (c) $n = 7$ *and G is* $\mathrm{AGL}(1,7)$;
  (d) $n = 8$ *and G is* $\mathrm{PGL}(2,7)$;
  (e) $n = 9$ *and G is* $\mathrm{PSL}(2,8)$ *or* $\mathrm{P\Gamma L}(2,8)$;
  (f) *G is $\mathcal{A}_n$ or $\mathcal{S}_n$.*

These results leave us with the third problem. Given $a \in \mathcal{T}_n \setminus \mathcal{S}_n$, we say that a group $G \leqslant \mathcal{S}_n$ is *a-normalizing* if

$$\langle a, G \rangle \setminus G = \langle g^{-1}ag \mid g \in G \rangle.$$

If G is a-normalizing for all $a \in \mathcal{T}_n \setminus \mathcal{S}_n$, then we say that G is *normalizing*. Recall that the *rank* of a transformation f is just the number of points in its image; we denote this by $\mathrm{rank}(f)$. For