Journal of Algebra 373 (2013) 491–501

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Coclass theory for nilpotent semigroups via their associated algebras

Andreas Distler[a,1], Bettina Eick[b,*]

[a] Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

[b] Institut Computational Mathematics, TU Braunschweig, Pockelsstrasse 14, 38106 Braunschweig, Germany

ARTICLE INFO

Article history:
Received 21 May 2012
Available online 25 October 2012
Communicated by Eamonn O'Brien

Keywords:
Coclass theory
Semigroups
Associative algebras

ABSTRACT

Coclass theory has been a highly successful approach towards the investigation and classification of finite nilpotent groups. Here we suggest a variation of this approach for finite nilpotent semigroups: we propose to study coclass graphs for the contracted semigroup algebras of nilpotent semigroups. We exhibit a series of conjectures on the shape of these coclass graphs. If these are proven, then this reduces the classification of nilpotent semigroups of a fixed coclass to a finite calculation. We show that our conjectures are supported by the nilpotent semigroups of coclass 0 and 1. Computational experiments suggest that the conjectures also hold for the nilpotent semigroups of coclass 2 and 3.

1. Introduction

A semigroup O or an associative algebra O is *nilpotent* if there exists an integer c so that every product of $c+1$ elements equals zero. The least integer c with this property is the *class* $\mathrm{cl}(O)$ of O; equivalently, the class of O is the length of series of powers

$$O > O^2 > \cdots > O^c > O^{c+1} = \{0\}.$$

The *coclass* of a finite nilpotent semigroup O with n non-zero elements or a finite dimensional nilpotent algebra O of dimension n is defined as $\mathrm{cc}(O) = n - \mathrm{cl}(O)$.

* Corresponding author.
E-mail addresses: adistler@fc.ul.pt (A. Distler), beick@tu-bs.de (B. Eick).

[1] The first author is supported by the project PTDC/MAT/101993/2008 of Centro de Álgebra da Universidade de Lisboa, financed by FCT and FEDER.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.09.042

For a semigroup S and a field K we denote by $K[S]$ the semigroup algebra defined by K and S. This is an associative algebra of dimension $|S|$. If S has a zero element z, then the subspace U of $K[S]$ generated by z is an ideal in $K[S]$. We call $K[S]/U$ the *contracted semigroup algebra* defined by K and S and denote it by KS. If S is a finite nilpotent semigroup, then KS is a nilpotent algebra of the same class and coclass as S.

Our first aim in this note is to suggest a general approach towards a classification up to isomorphism of nilpotent semigroups of a fixed coclass. For this purpose we choose an arbitrary field K and we define a directed labelled graph $\mathcal{G}_{r,K}$ as follows: the vertices of $\mathcal{G}_{r,K}$ correspond one-to-one to the isomorphism types of algebras KS for the nilpotent semigroups S of coclass r; two vertices A and B are adjoined by a directed edge $A \to B$ if $B/B^c \cong A$, where c is the class of B; each vertex A in $\mathcal{G}_{r,K}$ is labelled by the number of isomorphism types of semigroups S of coclass r with $A \cong KS$. Illustrations of parts of such graphs appear as Fig. 1 on page 498 and as Fig. 2 on page 500.

We have investigated various of the graphs $\mathcal{G}_{r,K}$ and we observed that all these graphs share the same general features. We formulate a sequence of conjectures and theorems describing these features. If our conjectures are proved, then this would provide a new approach towards the classification and investigation of nilpotent semigroups by coclass. In particular, it would show how the classification of the infinitely many nilpotent semigroups of a fixed coclass reduces to a finite calculation.

As a second aim, we exhibit some graphs $\mathcal{G}_{r,K}$ explicitly to illustrate our conjectures. We have determined the graphs $\mathcal{G}_{0,K}$ and $\mathcal{G}_{1,K}$ for all fields K using the classification of the nilpotent semigroups of small coclass in [2,3]; see Sections 5 and 6. Further, we investigated the graphs $\mathcal{G}_{2,K}$ and $\mathcal{G}_{3,K}$ for some finite fields K using computational methods based on [6] to solve the isomorphism problem for nilpotent associative algebras over finite fields; see Section 7.

One can also define a directed graph $\mathcal{G}_r$ whose vertices correspond one-to-one to the isomorphism types of semigroups of coclass r. While the graphs $\mathcal{G}_r$ are also of interest, they do not exhibit the same general features as $\mathcal{G}_{r,K}$. We compare $\mathcal{G}_r$ and $\mathcal{G}_{r,K}$ briefly in Section 9.

The idea of using coclass for the classification of nilpotent algebraic objects was first introduced by Leedham-Green and Newman [10] for nilpotent groups. We refer to the book by Leedham-Green and McKay [9] for background and many details on the results in the group case. Various details of the approach taken here are similar to the concepts in group theory. In particular, the idea of searching for periodic patterns in coclass graphs as used below also arises in group theory; we refer to [5,7] for details. But a nilpotent semigroup is not a group and hence the coclass theories for groups and semigroups are independent.

2. Coclass conjectures for semigroups

In this section we investigate general features of the graph $\mathcal{G}_{r,K}$ for $r \in \mathbb{N}_0$ and arbitrary field K.

By construction, every connected component of $\mathcal{G}_{r,K}$ is a rooted tree. Using basic results on nilpotent semigroups (see [3, Lemma 2.1]) one readily shows that $2r$ is an upper bound for the dimension of a root (that is, the dimension of the corresponding algebra) in $\mathcal{G}_{r,K}$. Thus $\mathcal{G}_{r,K}$ consists of finitely many rooted trees. We call an infinite path in a rooted tree *maximal* if it starts at the root of the tree.

Conjecture I. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Then the graph $\mathcal{G}_{r,K}$ has only finitely many maximal infinite paths. The number of such paths depends on r but not on K.*

For an algebra A in $\mathcal{G}_{r,K}$ we denote by $\mathcal{T}(A)$ the subgraph of $\mathcal{G}_{r,K}$ consisting of all paths that start at A. This is a rooted tree with root A. We say that $\mathcal{T}(A)$ is a *coclass tree* if it contains a unique maximal infinite path. A coclass tree $\mathcal{T}(A)$ is *maximal* if either A is a root in $\mathcal{G}_{r,K}$ or the parent of A lies on more than one maximal infinite paths.

1 Remark. Conjecture I is equivalent to saying that $\mathcal{G}_{r,K}$ consists of finitely many maximal coclass trees and finitely many other vertices.