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Locally simple subalgebras of diagonal Lie algebras

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ABSTRACT

We describe, up to isomorphism, all locally simple subalgebras of any diagonal locally simple Lie algebra.

1. Introduction

A Lie algebra $\mathfrak{g}$ is *locally finite* if any finite subset S of $\mathfrak{g}$ is contained in a finite-dimensional Lie subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}$. If, for any S, $\mathfrak{g}(S)$ can be chosen simple (semisimple), $\mathfrak{g}$ is called *locally simple* (*semisimple*). In 1998, A. Baranov introduced the class of diagonal locally finite Lie algebras and established their general properties, see [B1,B2]. Moreover, an explicit description of the more special class of diagonal locally simple Lie algebras was obtained by A. Baranov and A. Zhilinskii in [BZ], where they classified diagonal direct limits of simple complex Lie algebras up to isomorphism. In the present paper we work with the latter class of Lie algebras, and throughout the paper a diagonal Lie algebra will be assumed locally simple. Particular examples of such algebras are the classical infinite-dimensional complex Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$, which can be defined as the unions $\bigcup_{i\in\mathbb{Z}_{>1}}\mathrm{sl}(i)$, $\bigcup_{i\in\mathbb{Z}_{>1}}\mathrm{o}(i)$, and $\bigcup_{i\in\mathbb{Z}_{>1}}\mathrm{sp}(2i)$, respectively, for any inclusions $\mathrm{sl}(i)\subset\mathrm{sl}(i+1)$, $\mathrm{o}(i)\subset\mathrm{o}(i+1)$, and $\mathrm{sp}(2i)\subset\mathrm{sp}(2i+2)$, $i>1$. Moreover, the latter Lie algebras are the only countable-dimensional finitary locally simple complex Lie algebras, see [B3,B4,BS].

The semisimple subalgebras of semisimple finite-dimensional complex Lie algebras were described by A. Malcev and E. Dynkin more than half a century ago [M,D]. Recently, I. Dimitrov and I. Penkov characterized all locally semisimple subalgebras of $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$ [DP]. The same problem

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is of interest for the more general class of diagonal Lie algebras. It makes sense to first restrict the problem to describing, up to isomorphism, all locally simple subalgebras of diagonal Lie algebras. The purpose of this paper is to present a solution of the latter problem.

2. Preliminaries

The base field is $\mathbb{C}$. We assume that all Lie algebras considered are finite dimensional or countable dimensional. When considering classical simple Lie algebras, we consider the three types A, C, and O, where O stands for both types B and D.

A classical simple Lie subalgebra $\mathfrak{g}_1$ of a finite-dimensional classical simple Lie algebra $\mathfrak{g}_2$ is called *diagonal* if there is an isomorphism of $\mathfrak{g}_1$-modules

$$V_2 \downarrow \mathfrak{g}_1 \cong \underbrace{V_1 \oplus \cdots \oplus V_1}_{l} \oplus \underbrace{V_1^* \oplus \cdots \oplus V_1^*}_{r} \oplus \underbrace{T_1 \oplus \cdots \oplus T_1}_{z},$$

where V_i is the natural $\mathfrak{g}_i$-module ($i = 1, 2$), V_1^* is the dual of V_1, and T_1 is the 1-dimensional trivial $\mathfrak{g}_1$-module. The triple (l, r, z) is called the *signature* of $\mathfrak{g}_1$ in $\mathfrak{g}_2$. An injective homomorphism $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ is *diagonal* if $\varepsilon(\mathfrak{g}_1)$ is a diagonal subalgebra of $\mathfrak{g}_2$. The *signature* of ε is by definition the signature of $\varepsilon(\mathfrak{g}_1)$ in $\mathfrak{g}_2$.

An *exhaustion*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$$

of a locally finite Lie algebra $\mathfrak{g}$ is a direct system of finite-dimensional Lie subalgebras of $\mathfrak{g}$ such that the direct limit Lie algebra $\varinjlim \mathfrak{g}_n$ is isomorphic to $\mathfrak{g}$. A locally simple Lie algebra $\mathfrak{s}$ is *diagonal* if it admits an exhaustion by simple subalgebras $\mathfrak{s}_i$ such that all inclusions $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ are diagonal.

The following result is due to A. Baranov.

Proposition 2.1. *Any locally simple subalgebra of a diagonal Lie algebra is diagonal.*

Proof. Let $\mathfrak{s}$ be a locally simple subalgebra of a diagonal Lie algebra $\mathfrak{s}'$. Corollary 5.11 in [B1] claims that a locally simple Lie algebra is diagonal if and only if it admits an injective homomorphism into a Lie algebra associated with some locally finite associative algebra. Hence $\mathfrak{s}'$ admits an injective homomorphism into a Lie algebra $\mathfrak{g}$ associated with some locally finite associative algebra. Then there is an injective homomorphism $\mathfrak{s} \to \mathfrak{s}' \to \mathfrak{g}$, so $\mathfrak{s}$ is diagonal. □

This result reduces the study of locally simple subalgebras of diagonal Lie algebras to the study of diagonal subalgebras.

Next we introduce the notion of index of a simple subalgebra in a simple Lie algebra. This notion goes back to E. Dynkin [D]. For a simple finite-dimensional Lie algebra $\mathfrak{g}$ we denote by $\langle\ ,\ \rangle_{\mathfrak{g}}$ the invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ normalized so that $\langle \alpha, \alpha \rangle_{\mathfrak{g}} = 2$ for any long root α of $\mathfrak{g}$. If $\varphi : \mathfrak{s} \to \mathfrak{g}$ is an injective homomorphism of simple Lie algebras, then $\langle x, y \rangle_{\varphi} := \langle \varphi(x), \varphi(y) \rangle_{\mathfrak{g}}$ is an invariant non-degenerate symmetric bilinear form on $\mathfrak{s}$. Consequently,

$$\langle x, y \rangle_{\varphi} = I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi) \langle x, y \rangle_{\mathfrak{s}}$$

for some scalar $I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi)$. By definition $I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi)$ is the *index* of $\mathfrak{s}$ in $\mathfrak{g}$. If φ is clear from the context, we will simply write $I_{\mathfrak{s}}^{\mathfrak{g}}$. If U is any finite-dimensional $\mathfrak{s}$-module, then the *index* $I_{\mathfrak{s}}(U)$ of U is defined as $I_{\mathfrak{s}}^{\mathrm{sl}(U)}$, where $\mathfrak{s}$ is mapped into $\mathrm{sl}(U)$ through the module U. The following properties of the index are established in [D].