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Defect of compactness in spaces of bounded variation ☆

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ABSTRACT

Defect of compactness for non-compact imbeddings of Banach spaces can be expressed in the form of a profile decomposition. Let X be a Banach space continuously imbedded into a Banach space Y, and let D be a group of linear isometric operators on X. A profile decomposition in X, relative to D and Y, for a bounded sequence $(x_k)_{k\in\mathbb{N}} \subset X$ is a sequence $(S_k)_{k\in\mathbb{N}}$, such that $(x_k - S_k)_{k\in\mathbb{N}}$ is a convergent sequence in Y, and, furthermore, S_k has the particular form $S_k = \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}$ with $g_k^{(n)} \in D$ and $w^{(n)} \in X$. This paper extends the profile decomposition proved by Solimini [10] for Sobolev spaces $\dot{H}^{1,p}(\mathbb{R}^N)$ with $1 < p < N$ to the non-reflexive case $p = 1$. Since existence of "concentration profiles" $w^{(n)}$ relies on weak-star compactness, and the space $\dot{H}^{1,1}$ is not a conjugate of a Banach space, we prove a corresponding result for a larger space of functions of bounded variation. The result extends also to spaces of bounded variation on Lie groups.

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1. Introduction

In presence of a compact imbedding of a reflexive Banach space X into another Banach space Y, Banach–Alaoglu theorem implies that any bounded sequence in X has a subsequence convergent in Y. If the imbedding $X \hookrightarrow Y$ is continuous but not compact, it may be possible to characterize a suitable subsequence as convergent in X once one subtracts a suitable "defect of compactness", which typically, for sequences of functions, isolates the singular behavior of the sequence. In broad sense this approach is known as *concentration compactness*, and in its more specific form, when the defect of compactness is expressed as a sum of elementary concentrations, is called profile decomposition. Profile decompositions were introduced by Michael Struwe in 1984 for particular class of sequences in Sobolev spaces.

Definition 1.1. Profile decomposition of a sequence (x_k) in a reflexive Banach space X, relative to a group D of isometries of X, is an asymptotic representation of x_k as a convergent sum $S_k = \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}$ with $g_k^{(n)} \in D$, $w^{(n)} \in X$, such that $g_k(x_k - S_k) \rightharpoonup 0$ for any sequence $(g_k) \subset D$. In the latter case one says that $x_k - S_k$ converges to zero D-weakly.

We refer the reader for motivation of profile decomposition as an extension of the Banach–Alaoglu theorem, and a proof of both via non-standard analysis, to Tao [12]. For *general* bounded sequences in Sobolev spaces $\dot{H}^{1,p}(\mathbb{R}^N)$, the profile decomposition, relative to the group of translations and dilations, was proved in [10], and the D-weak convergence of the remainder was identified as convergence in the Lorentz spaces $L^{p^*,q}$, $q > p$, where $p^* = \frac{pN}{N-p}$, and $1 < p < N$ (which includes L^{p^*} but excludes $L^{p^*,p}$). The result of [10] was later reproduced by Gérard [5] and Jaffard [6], who extended it to the case of fractional Sobolev spaces, but, on the other hand, gave a weaker form of remainder. For general Hilbert spaces, equipped with a non-compact group of isometries of particular type, existence of profile decomposition was proved in [9]. This, in turn, stimulated the search for new concentration mechanisms, i.e. different groups D, that yield profile decompositions in concrete functional spaces. In particular profile decompositions were proved with inhomogeneous dilations $j^{-1/2}u(z^j)$, $j \in \mathbb{N}$, with z^j denoting an integer power of a complex number, for problems in the Sobolev space $H_0^{1,2}(B)$ of the unit disk, related to the Trudinger–Moser functional; and with the action of the Galilean invariance, together with shifts and rescalings, involved in the loss of compactness in Strichartz imbeddings for the nonlinear Schrödinger equations. For a more comprehensive summary of known profile decompositions, including Besov and Triebel–Lizorkin spaces we refer the reader to a recent survey [13]. Profile decomposition in the general, uniformly convex and uniformly smooth, Banach space was proved recently in [11], which required to substitute weak convergence with Delta-convergence of T.C. Lim [7]. Not unlike [11], this paper studies profile decomposition by adapting the prior work on the topic to the mode of convergence of weak type which is pertinent in the new set-