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Trace ideal criteria for embeddings and composition operators on model spaces ☆

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ABSTRACT

Let K_ϑ be a model space generated by an inner function ϑ. We study the Schatten class membership of composition operators $C_\varphi : K_\vartheta \to H^2(\mathbb{D})$ with a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, and, more generally, of embeddings $I_\mu : K_\theta \to L^2(\mu)$ with a positive measure μ in $\bar{\mathbb{D}}$. In the case of one-component inner functions ϑ we show that the problem can be reduced to the study of natural extensions of I and C_φ to the Hardy–Smirnov space $E^2(D)$ in some domain $D \supset \mathbb{D}$. In particular, we obtain a characterization of Schatten membership of C_φ in terms of Nevanlinna counting function. By example this characterization does not hold true for general ϑ.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk and $\mathbb{T} = \{z : |z| = 1\}$ be the unit circle. A bounded analytic function ϑ in $\mathbb{D}$ is said to be *inner* if its non-tangential boundary

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values satisfy $|\vartheta| = 1$ almost everywhere on $\mathbb{T}$. We denote by $H^2(\mathbb{D})$ the Hardy space on $\mathbb{D}$ and by $K_\vartheta = H^2(\mathbb{D}) \ominus \vartheta H^2(\mathbb{D})$ the corresponding model space.

In this article two classes of operators are considered: embeddings $I_\mu : K_\vartheta \to L^2(\mu)$, where μ is a finite positive measure supported on $\overline{\mathbb{D}}$, and composition operators $C_\varphi : f \mapsto f \circ \varphi$ acting from K_ϑ into $H^2(\mathbb{D})$, where $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function. In fact, it is well-known that the latter type of operator may be considered as a special case of the former for a certain pullback measure μ_φ. We mention that embeddings of model spaces have been studied by a number of authors [1,2,4–7,29]; composition operators on Hardy (and more general) spaces is by now a classical subject – we refer the reader to [8,27] and to [25] for a description of the current state of the art and a history survey. In this article we study the composition operator acting from the model space K_ϑ into $H^2(\mathbb{D})$ thus emphasizing interaction between the boundary behavior of φ and the spectrum of the inner function ϑ. In such setting the problem has been considered in [21]. Our main goal is to understand when such embedding and composition operators belong to the Schatten trace ideal $\mathcal{S}_p$, $0 < p < \infty$.

The embedding operators on K_ϑ have proved easier to analyze when ϑ is a *one-component* inner function, see [1,2,4,5,29]. In particular, the Schatten ideal membership of I_μ has been characterized by Baranov [4] for one-component ϑ. In Section 3 we suggest a different approach to the problem. We return to the original ideas of Cohn [6] and extend embedding operators on K_ϑ to operators acting on the Hardy–Smirnov space $E^2(D)$ of a certain domain $D \supset \mathbb{D}$. This allows us to obtain a geometrical criterion for the inclusion of I_μ in $\mathcal{S}_p$. In particular we recover the aforementioned result in [4].

For composition operators C_φ we further refine our result to give trace ideal criteria in terms of the Nevanlinna counting function N_φ,

$$N_\varphi(z) = \sum_{\varphi(\zeta)=z} \log \frac{1}{|\zeta|}.$$

We combine the geometric approach with recent results [17,18] that clarify the connection between the Nevanlinna counting function N_φ and the measure μ_φ (see also [11]), in order to obtain the following characterization.

Theorem 4.2. *Let ϑ be a one-component inner function. The operator $C_\varphi : K_\vartheta \to H^2$ is in $\mathcal{S}_p$, $0 < p < \infty$, if and only if*

$$\int_{\mathbb{D}} \left(\frac{N_\varphi(z)(1 - |\vartheta(z)|)^2}{1 - |z|^2} \right)^{p/2} \left(\frac{1 - |\vartheta(z)|^2}{1 - |z|^2} \right)^2 dA < \infty.$$

The article is organized as follows. The next section contains preliminary information about one-component inner functions and the corresponding model spaces. In Section 3 we reduce the trace ideal problem of embedding operators on K_ϑ to a corresponding